Geometry of curves with exceptional secant planes

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I. Linear series and secant planes

In my thesis, I studied linear series on curves, and tried to understand their "secant-plane" behavior in families.

To be concrete: say $C \subset \mathbb{P}^s$ is a curve. The family of lines meeting C has expected codimension 1 in $\mathbb{G}(1,3)$. So, since $\mathbb{G}(1,3)$ is 4-dimensional, we'd expect C to have finitely many quadrisecant lines, and no quintisecants.

The Italians computed the expected number of quadrisecants to be

$$Q = \frac{1}{12}(d-2)(d-3)^2(d-4) - \frac{1}{2}g(d^2-7d+13-g).$$

Now step back: view C not via its embedding in $\mathbb{P}^3$, but rather as an abstract curve equipped with a 3-dimensional linear series (L, V), where L is a line bundle and $V \subset H^0(C, L)$ is a subspace of the complete series defined by L. Then

$$\overline{p_1 p_2 p_3 p_4} \text{ is a quadrisecant line}$$
$$\Leftrightarrow$$
$$\text{rk}(V \xrightarrow{\text{ev}} H^0(L/L(-p_1 - p_2 - p_3 - p_4))) = 2.$$

More generally, given any curve C equipped with a linear series (L, V), the image of C will have a d-secant $(d - r - 1)$-plane spanned by $p_1, \ldots, p_d$ whenever the evaluation map

$$V \xrightarrow{\text{ev}} H^0(L/L(-p_1 - \cdots - p_d))$$

has rank $(d - r)$.

Today we'll study secant-plane behavior of curves that vary in families, with an eye to studying effective divisors on the moduli space of curves associated with codimension-1 secant-plane behavior.

Remarks:

- 2 aspects to the study (qualitative and quantitative). Want to compute classes of effective divisors on $\overline{\mathcal{M}}_g$ corresponding to curves with linear series with secant planes in codimension 1. So need to solve enumerative problems involving 1-parameter families of curves. But enumerative calculations only hold significance provided that on a general curve, there is *no* codimension-1 secant-plane behavior.

- For the toy "quadrisecants" example, enumerative significance wasn't established until 1980's.

II. Effective divisors on the moduli space
Recall: $\overline{\mathcal{M}}_g$ is a $(3g-3)$-dimensional projective variety. It compactifies the space of smooth curves of genus g, by allowing curves to become slightly singular (*stable*).

The Picard group of $\overline{\mathcal{M}}_g$ is generated over $\mathbb{Q}$ by classes $\lambda, \delta_0, \ldots, \delta_{\lfloor g/2 \rfloor}$. When $g > 2$, the classes are linearly independent, while if $g = 2$, "Mumford's relation"

$$10\lambda - \delta_0 - 2\delta_1 = 0$$

holds.

Why should we care about effective divisors on the moduli space?

Motivation from birational geometry: Given $g \geq 2$, is $\overline{\mathcal{M}}_g$ of general type?

Harris and Mumford showed that for all g sufficiently large, $\overline{\mathcal{M}}_g$ is of general type, which was a surprise (for low genus, $\overline{\mathcal{M}}_g$ has very special geometry.)

The Harris–Mumford proof relies on the construction of effective divisors associated to curves that have linear series with special codimension-1 behavior.

Improvements due to Eisenbud–Harris involve studying a different class of effective divisors: the Brill–Noether (BN) and Petri divisors.

BN curves admit linear series (L, V) when the expected dimension of such series is $\rho = -1$. Petri curves are those for which the cup-product

$$V \otimes H^0(C, K \otimes L^{-1}) \to H^0(C, K)$$

fails to be injective.

Up to a positive rational multiple, BN has class

$$\mathrm{BN} = (g+3)\lambda - \frac{g+1}{6}\delta_0 - \sum_{i=1}^{\lfloor \frac{g}{2} \rfloor} i(g-i)\delta_i.$$

In particular, the ratio of its lambda-coefficient to its δ_0-coefficient equals $6 + \frac{12}{g+1}$.

It turns out that for an effective divisor class

$$a\lambda - \sum_{i=0}^{\lfloor g/2 \rfloor} b_i\delta_i,$$

the single most important invariant (from the birational p.o.v.) is a/b_0, the *slope*.

Effective divisors of minimal slope s_g determine extremal rays in the effective cone.

Slope conjecture (Harris–Morrison): $s_g = 6 + \frac{12}{g+1}$, and BN are the only effective divisors on $\overline{\mathcal{M}}_g$ of minimal slope.

The slope conjecture is false. Farkas–Khosla produced infinitely many counterexamples attached to curves verifying a codimension-1 syzygy condition.

Khosla showed that when $\rho = 0$, one can calculate divisor classes in Pic $\mathcal{G}^s_m$, and push them down to Pic $\overline{\mathcal{M}}_g$ via explicit formulas. Here $\mathcal{G}^s_m$ is the space of linear series on genus-g curves, which maps finitely onto $\overline{\mathcal{M}}_g$ when $\rho = 0$.

III. Secant-plane divisors on $\overline{\mathcal{M}}_g$

We consider divisors associated to curves with codimension-1 secant-plane behavior. Assuming $\rho = 0$ and $\mu = d - r(s+1-d+r) = -1$, we compute secant-plane divisor classes in Pic $\mathcal{G}^s_m$, then push forward to Pic $\overline{\mathcal{M}}_g$ using Khosla's formulas.

Divisor classes in Pic $\mathcal{G}^s_m$ are determined by their "values" on 1-parameter families of curves with linear series.

Consider a 1-parameter family of curves $\pi : \mathcal{X} \to B$ with smooth general fiber, and finitely many irreducible nodal special fibers.

$\mathcal{X}$ comes equipped with

- A line bundle $\mathcal{L}$ with degree m on every fiber

- A rank-$(s+1)$ vector bundle $\mathcal{V} = \pi_*\mathcal{L}$

$\mathcal{X}/B \to \mathbb{P}\mathcal{V}^*$ is a family of g^s_m's.

How many fibers in the family have d-secant $(d-r-1)$-planes? Want answer in terms of tautological invariants.

One could try computing the locus of points in $\mathcal{X}_B^d$ for which the evaluation map

$$\mathcal{V} \stackrel{\text{ev}}{\to} S^d(\mathcal{L})$$

fibered in

$$V \to H^0(L/L(-p_1 - \cdots - p_d))$$

has rank $(d-r)$.

Difficulties with this approach:

- Not clear that our prescription for d-secant planes makes sense when p_i is a node of a fiber of $\mathcal{X}$. But Ran $\Rightarrow$ patch by replacing fiber product with Hilbert scheme.

Ran $\Rightarrow$ pushforward of degeneracy locus of ev to B is expression in

$\alpha = \pi_*(c_1^2(\mathcal{L})), \beta = \pi_*(c_1(\mathcal{L}) \cdot \omega), \gamma = \pi_* \omega^2,$
$c = c_1(\mathcal{V}),$ and $\delta_0 =$ # of singular fibers in π.

- Ran's approach isn't computationally practical (takes place over d copies of $\mathcal{X}$).

Alternative: use test families to deduce relations among tautological coefficients.

Test families:

1. Projections of general curve of degree m in $\mathbb{P}^{s+1}$ from points along disjoint line l: # of interesting fibers = # of d-secant $(d-r)$-planes to the g_m^{s+1} that intersect l.

2. Projections of general curve of degree $(m+1)$ in $\mathbb{P}^{s+1}$ from points along the curve: # of interesting fibers
$= (d+1) \times$ (# of $(d+1)$-secant $(d-r)$-planes to a g^{s+1}_{m+1}).

3. Fix a $K3$ surface $X \subset \mathbb{P}^s$ with Picard number 2 that contains a smooth curve C of degree m and genus g; take a generic pencil of curves of class $[C]$ on X.

    - Knutsen $\Rightarrow$ such $K3$ surfaces exist, and $r = 1 \Rightarrow$ none of these surfaces have d-secant $(d-r-1)$-planes.

    - If $r = s$, then none of these surfaces have d-secant $(d-r-1)$-planes, by Bézout's theorem.

Need 2 more relations. Get 1 more because formula is stable under renormalizing $c_1(\mathcal{L})$ by factors from B. Choose the renormalization that trivializes $\mathcal{V}$:

$$c_1(\mathcal{L}) \mapsto c_1(\mathcal{L}) - \frac{\pi^* c_1(\mathcal{V})}{s+1},$$
$$c_1(\mathcal{V}) \mapsto c_1\left(\mathcal{V} \otimes \mathcal{O}\left(-\frac{c_1(\mathcal{V})}{s+1}\right)\right) = 0.$$

If $r = 1$ or $r = s$, empirically deduce the missing apparent relation:

- $r = 1$:

  $$2(d-1)P_\alpha + (m-3)P_\beta = (6-3g)(P_\gamma + P_{\delta_0})$$

- $r = s$:

  $$2(s-1)P_\alpha + (2m-3s)P_\beta =$$
  $$(6s-3m)P_\gamma - (15m-30s+12-6g)P_{\delta_0}.$$

The case $r = 1$

When $r = 1$, we can go further, and determine (conjecturally) generating functions for the tautological coefficients P.

First determine a generating function for

$N_d(m) =$# of d-secant $(d-2)$-planes to a general curve of degree m in $\mathbb{P}^{2d-2}$.

Note that # of interesting fibers in the first test family $= N_d(m)$,
# of interesting fibers in the second family $= N_{d+1}(m+1)$.

Theorem 2:

$$\sum_{d \geq 0} N_d z^d = \left(\frac{2}{(1+4z)^{1/2}+1} \right)^{2g-2-m} \cdot (1+4z)^{\frac{g-1}{2}}.$$

Lehn's work $\Rightarrow$ such a formula should exist.

Ingredients of proof: Porteous' formula, combinatorics involving subgraphs of the complete graph on d labeled vertices, the "classical" formula for N_d recorded in [ACGH].

Now let

$$Z_m(z) := \left(\frac{2}{(1+4z)^{1/2}+1}\right)^{2g-2-m} \cdot (1+4z)^{\frac{g-1}{2}}.$$

Theorem 2, together with our relations among tautological coefficients, implies that

$$\sum_{d\geq 0} P_c(d,m)z^d = -Z_m(z),$$

$$\sum_{d\geq 0} P_\alpha(d,m)z^d = Z_m(z)\cdot\left[\frac{1}{2}-\frac{1}{2(1+4z)^{1/2}}\right]$$

$$\sum_{d\geq 0} P_\beta(d,m)z^d = Z_m(z)\cdot\left[\frac{2z}{1+4z}-\frac{4z}{(1+4z)^{1/2}((1+4z)^{1/2}+1)}\right],$$

and conjecturally also

$$\sum_{d\geq 0} P_{\gamma}(d,m)z^d = Z_m(z)\cdot\left[\frac{z(32z^2-7(1+4z)^{3/2}+36z+7)}{6(1+4z)^{5/2}((1+4z)^{1/2}+1)}\right] \text{ and}$$

$$\sum_{d\geq 0} P_{\delta_0}(d,m)z^d = Z_m(z)\cdot\left[\frac{z(32z^2-(1+4z)^{3/2}+12z+1)}{6(1+4z)^{5/2}((1+4z)^{1/2}+1)}\right].$$

Finally, let

$$X(z) := \frac{z(32z^2-7(1+4z)^{3/2}+36z+7)}{6(1+4z)^{5/2}((1+4z)^{1/2}+1)}, \text{ and}$$

$$Y(z) := \frac{z(32z^2-(1+4z)^{3/2}+12z+1)}{6(1+4z)^{5/2}((1+4z)^{1/2}+1)}.$$

Reduction: ETS $X(z)$ and $Y(z)$ are exponential generating functions for constant terms of $P_{\gamma}(d,m)$ and $P_{\delta_0}(d,m)$, respectively, viewed (for fixed choices of d) as polynomials in m and $(2g-2)$.

$Y(z)$ has Taylor series

$$\frac{1}{6}(3z^2 - 20z^3 + 105z^4 - 504z^5 + 2310z^6 - 10296z^7 + \dots);$$

$$[z^n]Y(z) = \frac{(-1)^{n-2}}{6} \cdot \frac{(2n-1)!}{n!(n-2)!}.$$

$X(z)$ has Taylor series

$$\frac{1}{6}(-3z^2 + 28z^3 - 177z^4 + 960z^5 - 4806z^6 + 22920z^7 - \dots);$$

$$[z^n]X(z) = (-1)^{n-1}\left(4^{n-1}\sum_{i=1}^{n-1}\frac{\binom{2i}{i-1}}{4^i} - \frac{1}{6} \cdot \frac{(2n-1)!}{n!(n-2)!}\right).$$

To prove the reduction, ETS $X(z)$ = exponential generating function for:

$S(d) :=$ weighted # of connected $(d+1)$-edged subgraphs of the complete graph on d labeled vertices $v_1, \dots, v_d$

where edges have multiplicity ≤ 3, and each graph $\mathcal{G}$ is assigned weight

$$w_{\mathcal{G}} := \underbrace{\prod_{i=2}^{d-1} \binom{\text{indeg}(v_i)}{m_{j_1,i}, \dots, m_{j_k,i}}}_{\text{multiplicities of edges incident to and pointing towards } v_i}.$$

Examples of secant-plane divisors

- $r = 1, d = 2, s = 3$. In this case, Sec $\subset \mathcal{G}^3_m$ comprises 3-dimensional linear series with double points. We have

$$2!\text{Sec} = (-6 + 2m)\alpha - 4\beta + (2g - 2 + 3m - m^2)c - \gamma + \delta_0.$$

- $r = 1, d = 3, s = 5$ **(case of 5-dimensional series with trisecant lines).** We have

$$3!\text{Sec} = (3m^2 - 27m - 6g + 66)\alpha + (72 - 12m)\beta + (28 - 3m)\gamma$$
$$+ (3m - 20)\delta_0 + (24 - m^3 + 9m^2 + 6mg - 26m - 24g)c.$$

- $r = 1, d = 4, s = 7$ **(case of 7-dimensional series with 4-secant 2-planes).** We have

$$4!\text{Sec} = (-1008 + 168g - 24mg - 72m^2 + 452m + 4m^3)\alpha$$
$$+ (360m - 1440 + 48g - 24m^2)\beta + (12g - 720 + 130m - 6m^2)\gamma$$
$$+ (372g - 360 + 342m - 119m^2 - m^4 + 18m^3 - 12g^2 - 132mg$$
$$+ 12m^2g)c + (6m^2 - 98m - 12g + 432)\delta_0.$$

- $r = 1, d = 5, s = 9$ **(case of 9-dimensional series with 5-secant 3-planes).** We have

$$5!\text{Sec} = (1020mg - 60m^2g - 4500g + 60g^2 + 19560 + 5m^4$$
$$+ 1735m^2 - 150m^3 - 9270m)\alpha$$
$$+ (240mg - 2400g + 33600 - 40m^3 - 10160m + 1080m^2)\beta$$
$$+ (20000 + 60mg - 800g + 370m^2 - 10m^3 - 4640m)\gamma$$
$$+ (20m^3g - 60mg^2 - 420m^2g + 6720 + 480g^2 + 2980mg$$
$$- 5944m + 30m^4 - 355m^3 + 2070m^2 - m^5 - 7200g)c$$
$$+ (60mg + 640g + 10m^3 + 2960m - 290m^2 - 10720)\delta_0.$$

IV. Slope asymptotics

If $\rho = 0, \mu = -1,$ and $r = 1$, then

$$g = 2ad \text{ and } m = (a+1)(2d-1), a \geq 2.$$

Then our virtual slope $\frac{b_\lambda}{b_0}$ satisfies

$$\begin{aligned}\frac{b_\lambda}{b_0} - \left(6 + \frac{12}{2ad+1}\right) &= \frac{3}{ad(a+1)} + O(d^{-2}) \\ &= \frac{6}{(a+1)g} + O(g^{-2}).\end{aligned}$$