1 Polytopes historical background

"A tendency in mathematics to greater and greater abstractness should not lead us to abandon our roots. In studying abstract polytopes, we shall always bear in mind the geometric origins of the subject" P. McMullen and E. Schulte.

4500 years ago. Egypt pyramids: are they half of an octahedron?

4000 years ago. Stones carved in polyhedra shapes showing all the symmetry groups and even some duality relations! This stones had been found in Scotland.

2500 years ago. Etruscans used dices in forms of the cube and the dodecahedron, probably for gambling.

Early Greeks. The Pithagoreans started studying the regular convex polygons and the pentagram. The discovery of the five regular solids has been attributed to Pythagoras (582-500BCE) by Eudemus, however, they were named after Plato by Heron.

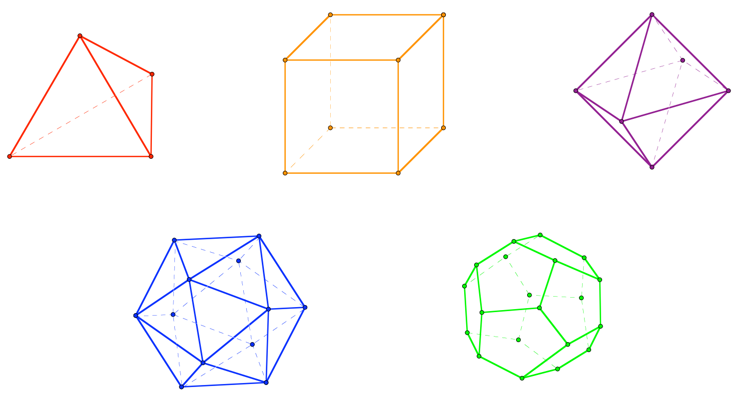

Figure 1: The five Platonic Solids.

Euclid demonstrates in his *Elements* that there are only five regular solids.

In this context, a convex polygon is regular if it has all sides of the same length and all its angles measure the same. A solid (or convex polyhedron) is regular if all its faces are congruent regular polygons and with the same number of them arranged around each vertex.

Exercise 1 *Show that there exist at most five regular solids.*

Archimedes (287-212 BCE) used the regular 96-gon to find that $3\frac{10}{71} < \pi < 3\frac{1}{7}$ and study the so-called *Archimedean solids* (or convex uniform polyhedra).

An Archimedean solid is a convex polyhedron with two or more types of regular polygons as faces, meeting in identical vertices. There are 13 of them.

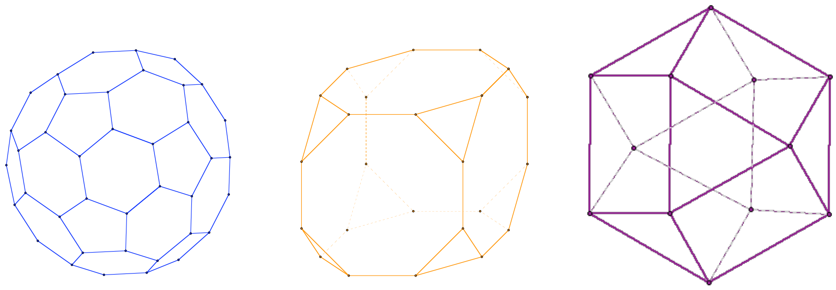

Figure 2: Three of the Archimedean solids.

Exercise 2 *What is the difference between the two polyhedra in Figure 3? Would you say that they are Archimedean solids? Why?*

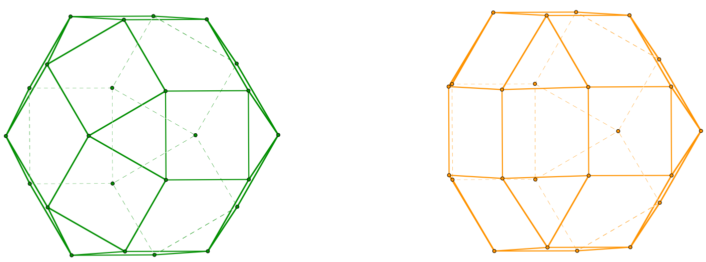

Figure 3: Find the differences.

Star-polytopes Thomas Bradwardine (1290-1349) systematically investigated the star-polygons $\{\frac{n}{d}\}$. The first appearance of the star-polyhedra are attributed to Paolo Uccello (1397-1475) and Wenzel Jamnitzel (1508-85). Each of them draw one of the star-polyhedra, but they did not seem to have studied them.

Johannes Kepler (1571-1630) re-discover Uccello's $\{\frac{5}{2}, 3\}$ and discover the $\{\frac{5}{2}, 5\}$. In 1809, Louis Poinsot (1777-1859) re-discover these two star-polyhedra, and found their duals. In 1811 Augustin Louis Cauchy (1789-1857) showed that these four are the only 'regular' star-polyhedra.

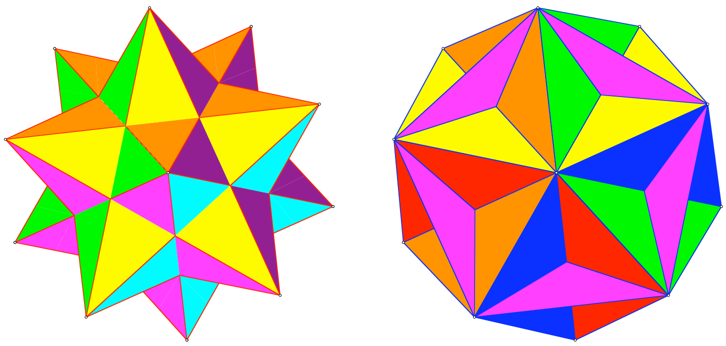

Figure 4: Two of the four Kepler-Poinsot polyhedra

Schläfli (1814-95) was one of the first mathematicians working on the concept of higher dimensions. Around 1850 he discover regular polytopes and honeycombs in four dimensions.

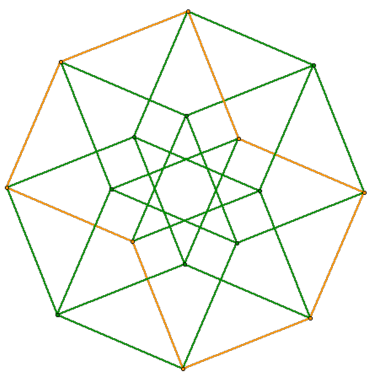

Figure 5: The hypercube.

Coxeter (1907-2003) set the direction towards the later developments of the theory of polytopes. In 1948 he publish the first edition of his *Regular Polytopes* in which he consolidates the study of regular polytopes in higher dimensions. He also consider 'regular maps' as polyhedra and studies their groups of symmetries.

Modern days. In the early 1920's Petrie discovered two infinite regular polyhedra in the Euclidean three space, allowing the 'vertex-figures' to be skew polygons. Coxeter then find a third one and showed that the enumeration was complete.

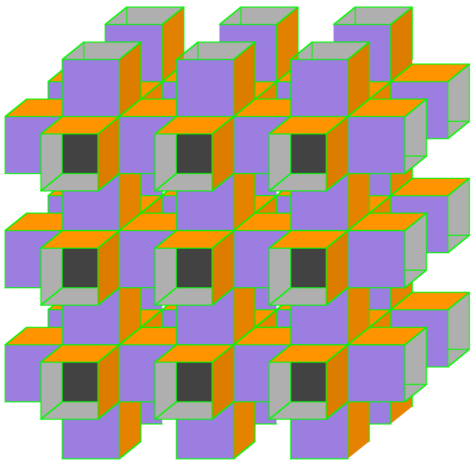

Figure 6: Petrie's infinite $\{4, 6|4\}$.

In the theory of polytopes, facets and vertex-figures were consider to be spherical until around 1975, Grünbaum took Coxeter's approach to polytopes even further and suggested the study of a larger class of objects (that he called polystroma), in which facets and vertex-figures might not be spherical. In the earlies 1980's, Danzer and Schulte extended this further and set out the basic theory of combinatorial objects that now we know as abstract polytopes.

Though out history, the main focus in the study of polytopes, has been on those with a 'large' group of symmetries.

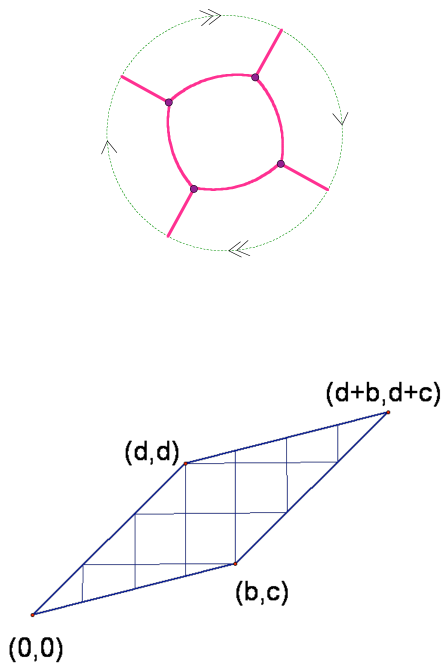

Figure 7: Examples of polyhedra that do not live in the Euclidean three Space.

2 Groups and actions

A set G, together with a binary operation $*$ is said to be a *group* if given $g_1, g_2, g_3 \in G$,

- $g_1 * g_2 \in G$.
- $(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$.
- There exists an element $\epsilon \in G$ such that for every element $g \in G$, $\epsilon * g = g = g * \epsilon$.
- For every $g \in G$ there exists $g^{-1} \in G$ such that $g * g^{-1} = \epsilon = g^{-1} * g$.

Exercise 3 *Which of the following are groups?*

1. $(\mathbb{Z}^n, +)$
2. $(\mathbb{Z}^n, \times)$
3. $(\mathbb{R}, +)$
4. $(\mathbb{R}, \times)$

5. $(\mathbb{R}^*, \times)$

Exercise 4 *Given the set* $\mathbf{n} := \{1, 2, \ldots, n\}$*, a* permutation *of* $\mathbf{n}$ *is a one-to-one function from* $\mathbf{n}$ *onto itself. Let* S_n *denote the set of all permutations of* $\mathbf{n}$*. Then* S_n *is a group, under the composition of permutations.*

The *action* of a group G on a set Z is an operation $\cdot : Z \times G \to Z$, such that $z \cdot \epsilon = z$ and $(z \cdot g) \cdot h = z \cdot (gh)$, for every $g, h \in G$ and $z \in Z$.

3 Isometry groups

"God is always doing geometry"
Plato

An *isometry* of $\mathbb{E}^n$ is a point to point transformation of the space onto itself that preserves distances. The product of two isometries is also an isometry and every isometry has an inverse. Hence, the set of all isometries of $\mathbb{E}^n$ forms a group, where the identity is the isometry that fixes every point.

Consider an n-gon in the Euclidean plane, that is a polygon with n sides. A *symmetry* of the n-gon is an isometry of the plane that sends the n-gon to itself. A convex n-gon is said to be regular if the n sides are of equal length and the angle between any two adjacent sides is always the same.

Exercise 5 *How many symmetries are there for a regular* n*-gon? How can we describe them?*

Given a polyhedron $\mathcal{P}$ in the Euclidean three space, a *symmetry* T of $\mathcal{P}$ is an isometry of $\mathbb{E}^3$ that sends $\mathcal{P}$ to itself. In particular, T sends vertices to vertices, edges to edges and faces to faces. Hence we can think that the group of symmetries of $\mathcal{P}$ acts on the vertices, edges and faces of $\mathcal{P}$.

Exercise 6 *What is the main difference of the action of the symmetry groups of the polyhedra in Figure 3?*

3.1 Isometries of the plane

In this section by an isometry we shall mean an isometry of the Euclidean plane.

We shall denote a reflection on a line a as ρ_a. A *glide refection* is the product of three reflections $\rho_a\rho_b\rho_c$, where the lines a and b are parallel lines, and c is a line perpendicular to both a and b.

Lemma 7 *If an isometry fixes two point on a line, then it fixes that line point-wise. If an isometry fixes three non-colinear points, then it is the identity.*

Corollary 8 *Let P, Q, R be three non-collinear points, and let α, β be two isometries such that $P\alpha = P\beta$, $Q\alpha = Q\beta$ and $R\alpha = R\beta$. Then $\alpha = \beta$.*

Lemma 9 *An isometry that fixes two points is a reflection or the identity.*

Lemma 10 *An isometry that fixes exactly one point is a product of two reflections.*

Corollary 11 *An isometry that fixes a point is a product of at most two reflections.*

Theorem 12 *Every isometry is a product of at most three reflections.*

Proposition 13 *A product of two reflections in parallel lines is a translation; and every translation is the product of two reflections in parallel lines.*

Proposition 14 *A product of two reflections in intersecting lines is a translation; and every translation is the product of two reflections in intersecting lines.*

Exercise 15 *Describe the possible isometries that are a product of exactly three reflections.*

Exercise 16 *Let P, Q, P', Q' be point on the plane, and let $|PQ|$ denote the distance between P and Q.*

a) *If $|PQ| = |P'Q'|$, then there exists an isometry α, which is either a translation or a rotation, such that $P\alpha = P'$ and $Q\alpha = Q'$.*

b) *If $|PQ| = |P'Q'|$, then there exists an isometry β, which is either a reflection or a glide reflection, such that $P\beta = P'$ and $Q\beta = Q'$.*

c) *Suppose that P, Q, P', Q' are point satisfying $|PQ| = |P'Q'| \neq 0$. Then there are exactly two isometries sending P to P' and Q to Q'; one of each of the above kinds.*

d) ***Theorem 17** (M. Chasles 1793-1880) Every isometry of the plane is a reflection, a translation, a rotation or a glide relfection.*

3.2 Isometries of the space and symmetry groups of polyhedra

For now, let us think that a polyhedron is a solid figure bounded by plane polygons in $\mathbb{E}^3$ (i.e., we are considering only convex polyhedra). As we pointed out before, there are five regular such polyhedra, the platonic Solids, that the Greeks identified with the four elements, earth, air, fire and water, and the whole universe.

If Π is a plane, then the *reflection* ρ_Π is the isometry of the space that fixes all the point on Π, and for every $P \notin \Pi$, the plane Π is the perpendicular bisector of the segment line $P\Pi(P)$. If Δ is a perpendicular plane to Π, then $\rho_\Pi\rho_\Delta$ is a *translation* along the common perpendicular lines to the planes. If Δ and Π are two planes that intersect at a line l, then $\rho_\Pi\rho_\Delta$ is a *rotation* about the axis l. Given a plane Γ that is perpendicular to intersecting planes Π and Δ, $\rho_\Pi\rho_\Delta\rho_\Gamma$ is a *rotatory reflection.*

Exercise 18 *List all the rotations of a tetrahedron as permutations of the four vertices. List the other 12 symmetries of the tetrahedron. Which of these are given by reflection in a plane? Show that those that are not reflections can be described as rotatory reflections.*

Exercise 19 a) *Mark the vertices of an octahedron $1, 2, \ldots, 6$. List all the rotations of the octahedron by the permutations they induce on the vertices. How many elements of each kind are there? What are their orders? How many in all?*

b) The octahedron has four axes, a, b, c, d running through the centres of opposite faces. Any rotation induces a permutation of a, b, c, d. Thus we get a map $\psi : R \to S_4$ from the set of rotations to the symmetric group on the four letters a, b, c, d. Show that R has at least 24 elemets, show that the map ψ is injective, and conclude that R is a group isomorphic to S_4.

c) Find subgroups of the group of rotations of the octahedron isomorphic to $\mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{D}_2, \mathcal{D}_3 = S_3, \mathcal{D}_4$, and describe them in terms of the geometry of the octahedron. (Where $\mathcal{D}_n$ denotes the dihedral group *of order $2n$, that is, the group of symmetries of a regular n-gon, and $\mathcal{C}_n$ denotes the* cyclic group *of order n, that is also the rotational subgroup of a regular n-gon.)*

d) Show that the group of all symmetries of the octahedron is a group of order 48.

Proposition 20 *In the Euclidean three space we have that:*

a) If an isometry fixes two point on a line, then it fixes that line point-wise. If an isometry fixes three non-colinear points, then it fixes the plane through those points, point-wise. If an isometry fixes four non-coplanar points, then it is the identity.

b) An isometry is completely determined by the image of four non-coplanar points.

c) An isometry that fixes three non-collinear points is either the identity or a reflection.

d) An isometry that fixes exactly one line point-wise is a rotation.

e) An isometry that fixes exactly one point is a rotatory reflection.

f) An isometry that fixes at least one point is the product of at most three reflections.

g) Every isometry is the product of at most four reflections.

Exercise 21 *Find all the symmetries of the icosahedron. How many symmetries of each kind are there? How many in total?*

Given a line l, a *glide rotation* or a *screw* is a translation along l, followed by a rotation about l. Given a plane Γ perpendicular to parallel planes Π and Δ, $\rho_\Pi \rho_\Delta \rho_\Gamma$ is a *glide reflection* with axis Γ.

Theorem 22 *Every isometry of the Euclidean three space is a reflection, a translation, a rotation, a rotatory reflection, a glide rotation or a glide reflection.*

3.2.1 Finite groups of isometries

Lemma 23 *Every finite group of isometries leaves at least one point invariant.*

Proposition 24 *The only finite groups of rotations in three dimensions are the cyclic groups $\mathcal{C}_n$, $n = 1, 2, \ldots$, the dihedral groups $\mathcal{D}_n$, $n = 2, 3, \ldots$, the tetrahedral group $\mathcal{T}$ (isomorphic to A_4), the octahedral group $\mathcal{O}$ (isomorphic to S_4) and the icosahedral group $\mathcal{I}$ (isomorphic to A_5).*

Exercise 25 *Find all the finite groups of symmetries in three dimensions.*

4 Convex polytopes

An *affine subspace* A of the Euclidean n-space is a subset which contains each line $\{(1-\lambda)a + \lambda b \mid \lambda \in \mathbb{R}\}$ between any two points $a, b \in A$. A *hyperplane* of $\mathbb{E}^n$ is an affine subspace of dimension $n-1$. The *affine hull* of a set $S \subset \mathbb{E}^n$, denoted by $Aff(S)$ is the intersection of all the affine subspaces of $\mathbb{E}^n$ which contains S.

In terms of linear algebra, a hyperplane is a set

$$\mathcal{H} := \{(x_1, x_2, \ldots x_n) \in \mathbb{E}^n \mid a_1x_1 + a_2x_2 + \ldots + a_nx_n = \alpha\},$$

where $a_1, \ldots, a_n, \alpha \in \mathbb{R}$ are fixed real numbers. In other words, each hyperplane of $\mathbb{E}^n$ is determined by a direction $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$, and a real number α; hence sometime we shall write $\mathcal{H}$ as $\mathcal{H}^a_\alpha$. Each hyperspace defines two *halfspaces*, namely,

$$\mathcal{H}^+ := \{(x_1, x_2, \ldots x_n) \in \mathbb{E}^n \mid a_1x_1 + a_2x_2 + \ldots + a_nx_n \geq \alpha\},$$
$$\mathcal{H}^- := \{(x_1, x_2, \ldots x_n) \in \mathbb{E}^n \mid a_1x_1 + a_2x_2 + \ldots + a_nx_n \leq \alpha\}.$$

A subset K of the Euclidean n-space is said to be *convex* if for any two points $a, b \in K$, the segment line $[a, b] := \{(1-\lambda)a + \lambda b \mid 0 \leq \lambda \leq 1\}$ is contained in K.

Exercise 26 *Show that the intersection of finitely many halfspaces in $\mathbb{E}^n$ is a convex set.*

Note that the intersection of two convex sets is again convex. The *convex hull* of a set $S \subset \mathbb{E}^n$, denoted by $Conv(K)$ is the intersection of all the convex sets that contain S.

Exercise 27 *Show that the convex hull of a set S is the set*

$$\left\{ \sum_{i=1}^{t} \lambda_i p_i \mid p_i \in S, \lambda_i \in \mathbb{R}, \lambda_i \geq 0, \sum_{i=1}^{t} \lambda_i = 1 \right\}.$$

The convex hull of a finite number of points in $\mathbb{E}^n$ is called a *convex polytope*. A convex polytope $\mathcal{P}$ is said to be a *k-polytope* if its affine hull is a subspace of dimension k.

Exercise 28 *Show that the set*

$$C_n := \{(x_1, x_2, \ldots, x_n) \in \mathbb{E}^n \mid 0 \leq x_i \leq 1 \text{ for every } i = 1, \ldots, n\}$$

is a convex set. Furthermore, show that it is a n-polytope. Can you express C_n as the intersection of halfspaces?

The empty set $\emptyset$ can be thought as the affine subspace of dimension -1, and it is also a polytope. A 0-polytope is a vertex and a 1-polytope is an edge. A *polygon* is a 2-polytope and a 3-polytope is called a *polyhedron*.

If the affine hull of points $p_0, p_1, \ldots p_n \in \mathbb{E}^n$ is of dimension n, then $Conv\{p_0, p_1, \ldots p_n\}$ is a convex n-polytope, called a *n-simplex*.

Given a n-polytope P, we can embed it in $\mathbb{E}^{n+1}$. If $p \in \mathbb{E}^{n+1}$ is such that $p \notin \mathrm{Aff}(P)$, then $Conv(P, p)$ is called a *pyramid* and it is a polytope.

Exercise 29 *Show that if P and P' are polytopes, then*

$$P \times P' := \{(a, a') \mid a \in P,\ a' \in P'\}$$

is again a polytope.

Theorem 30 *Every convex polytope is the intersection of a finite number of halfspaces.*

Given a convex set K and a hyperplane $\mathcal{H}_\alpha^a$, we say that $\mathcal{H}_\alpha^a$ *supports* K if

$$\alpha = \sup\{x_1 a_1 + x_2 a_2 + \ldots + x_n a_n \mid x \in K\}.$$

Given a polytope P and a hyperplane $\mathcal{H}_\alpha^a$ that supports P , the intersection $F := P \cap \mathcal{H}_\alpha^a$ is called a *proper face* of P. Note that every face F of P is also a convex polytope, hence it has a dimension. If F is a j-polytope, then we say that F is a j-face of P. The faces of dimensions $0, 1$ and $n-1$ are called *vertices, edges* and *facets*, respectively. Often we shall also say that $\emptyset$ and P are the *improper faces* of P. We denote by $\mathcal{P} := \mathcal{P}(P)$ to the set of all (proper and improper) faces of P.

We now turn our attention to some combinatorial properties of convex polytopes, which will then motivate our definition of an abstract polytope.

a) $\mathcal{P}$ is a partially ordered set (poset), under the order $F \leq G$ if and only if $F \subseteq G$. We shall say that two faces F and G are *incident* if $F \leq G$ or $G \leq G$

b) $\mathcal{P}$ is a lattice, where the *meet* of two faces F and G is $F \wedge G := F \cap G$, and the *join* $F \vee G$ is the unique smallest face of $\mathcal{P}$ that contains both F and G.

c) If $F < G$ are such that $\dim F = j-1$ and $\dim G = j+1$, there are exactly two faces H of dimension j such that $F < H < G$.

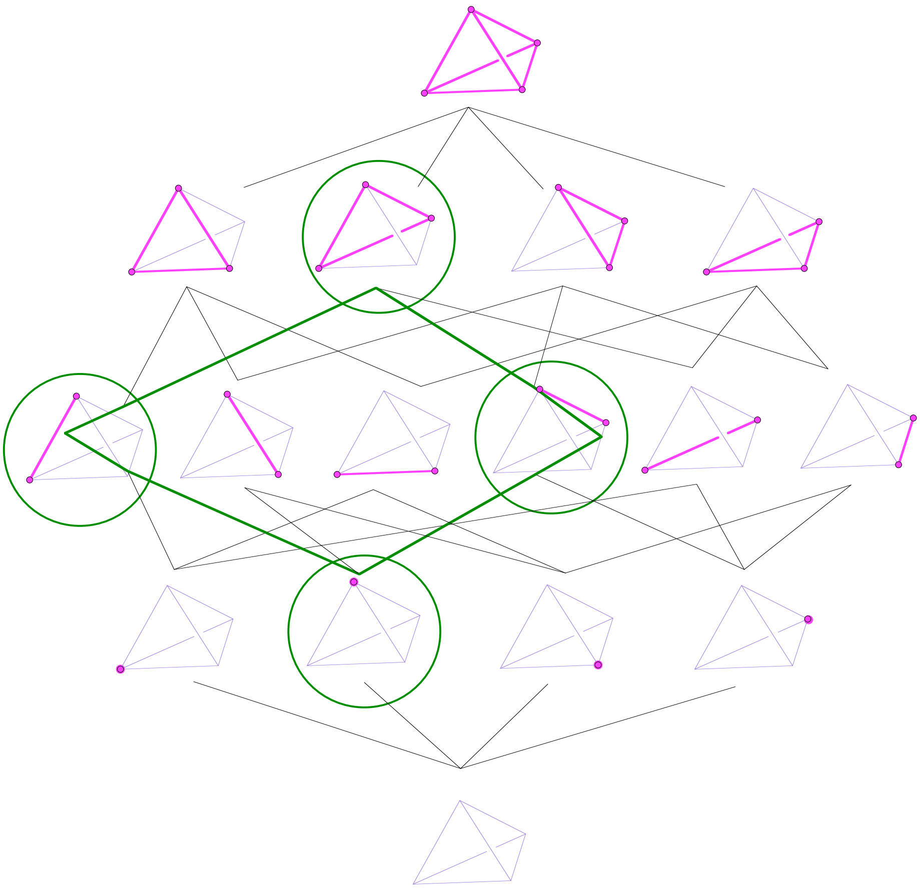

Figure 8: A tetrahedron, represented as a poset.

d) For every two faces $F, G \in \mathcal{P}$ such that $F \leq G$, the *section*

$$G/F := \{H \in \mathcal{P} \mid F \leq H \leq G\}$$

of $\mathcal{P}$ is isomorphic to the face-lattice of a polytope of dimension $\dim G - \dim F - 1$.

e) If $\dim P \geq 2$, the $\mathcal{P}$ is *connected* in the sense that any two proper faces can be joined by a chain of (proper) incident faces.

f) $\mathcal{P}$ is *strongly connected*, that is, if F and G are two proper faces of P such that $\dim G \geq \dim F + 3$, then the section G/F is connected.

4.1 Regular convex polytopes

A symmetry g of P is an isometry of $\mathbb{E}^n$ such that $Pg = P$, and the group of symmetries of P is denoted $G(P)$. An automorphism of P is a permutation of the face-lattice $\mathcal{P}$ which preserves the inclusion; the automorphism group of P is denoted $\mathrm{Aut}(P)$.

Exercise 31 *Given a convex polytope P, is every symmetry of P an automorphism? What about the other way around?*

A *flag* of a n-polytope P is a subset of pairwise incident faces of the form $\{F_{-1}, F_0, F_1, \ldots, F_{n-1}, F_n\}$ (where F_j is a j-face of P). The set of all flags of P is denoted by $\mathcal{F}(P)$.

A convex poytope P is said to be *regular* if the symmetry group $G(P)$ is transitive on the flags. It is said to be *combinatorially regular* if its automorphism group is transitive on the flags.

Exercise 32 *Is every regular polytope a combinatorially regular polytope? What about the other way around?*

Exercise 33 *Consider the two following definitions:*

- *A polygon is (combinatorially) regular if all its edges are of the same length and all the angles between them are the equal. For $n \geq 3$, a n-polytope is (combinatorially) regular if the facets are regular and congruent and the vertex-figures are isomorphic.*

- *A polytope is (combinatorially) regular if $G(P)$ $(\mathrm{Aut}(P))$ is transitive on j-faces, for every $j = 0, \ldots, n-1$.*

Are these two definitions equivalent? Are they equivalent to the ones given before for regular and combinatorially regular poytopes?

Note that to every involutory isometry R of $\mathbb{E}^n$, we can associate a *mirror* $\{x \in \mathbb{E}^n \mid xR = x\}$ (i.e. the set of all the points invariant under R). The mirror of a (hyperplane) reflection is a hyperplane!

A *Coxeter group* is one of the form $G := \langle R_0, R_1, \ldots, R_{n-1} \rangle$, where each R_i is an involution and satisfies (only!) the relations $R_iR_j)^{p_{ij}} = E$, the identity, where each p_{ij} is a positive integer or infinity, and $p_{ij} = p_{ji}$, $p_{ii} = 1$. A Coxeter group is called a *string* Coxeter group if $p_{ij} = 2$ if $|i-j| > 1$. Defining $p_j := p_{j-1,j}$, we denote the above string Coxeter group by $[p_1, \ldots, p_{n-1}]$.

Theorem 34 *The symmetry group of a regular convex n-polytope is a finite string Coxeter group. The generators of the group are (hyperplane) reflections $R_0, \ldots R_{n-1}$ and each $p_j \geq 3$.*

Theorem 35 *Any finite string Coxeter group for which $p_j \geq 3$, for every $j = 1, \ldots, n-1$ is (isomorphic to) the symmetry group of a regular convex polytope.*

5 Abstract polytopes

An *(abstract) polytope* of rank n or an *n-polytope* is a partially ordered set $\mathcal{P}$ endowed with a strictly monotone rank function having range $\{-1, \ldots, n\}$. For $0 \leq j < n$, the elements of $\mathcal{P}$ of rank j are called *j-faces*, and often a j-face is denoted by F_j. The faces of rank $0, 1$ and $n-1$ are usually called the *vertices*, *edges* and *facets* of the polytope, respectively. We require that $\mathcal{P}$ has a smallest face F_{-1}, and a greatest face F_n (called the *improper faces* of $\mathcal{P}$), and that each maximal chain (called a *flag*) of $\mathcal{P}$ contains exactly $n+2$ faces. We denote by $\mathcal{F}(\mathcal{P})$ the set of all flags of $\mathcal{P}$. Two flags are said to be *adjacent* if they differ by exactly one face. Also, we require that $\mathcal{P}$ be strongly flag-connected, that is, any two flags $\Phi, \Psi \in \mathcal{F}(\mathcal{P})$ can be joined by a sequence of flags $\Phi = \Phi_0, \Phi_1, \ldots, \Phi_k = \Psi$ such that each two successive flags Φ_{i-1} and Φ_i are adjacent, with $\Phi \cap \Psi \subseteq \Phi_i$ for all i. Finally, we require the homogeneity property (often called the *diamond condition*), that is, whenever $F \leq G$, with $\text{rank}(F) = j-1$ and $\text{rank}(G) = j+1$, there are exactly two faces H of rank j such that $F \leq H \leq G$.

Exercise 36 *Are the geometric structures of Figure 9 abstract polytopes?*

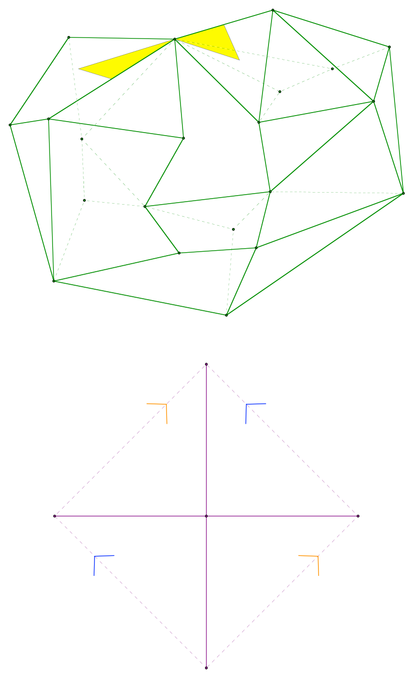

Figure 9: Are they polytopes?

A 0-polytope contains only two (incident) elements, F_{-1} and F_0; hence, up to isomorphism, there is only one 0-polytope, and it can be thought of as a single point or vertex. A 1-polytope must have a diagram with diamond shape (see Figure 10), and we can think of it as an edge with its two end-vertices.

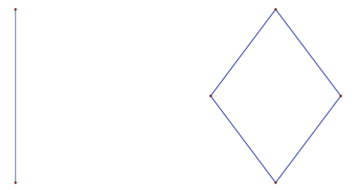

Figure 10: Rank 0 and 1 polytopes, respectively, as posets.

If $\mathcal{P}$ is a 2-polytope, it is easy to see that the number of vertices and edges of $\mathcal{P}$ are exactly the same. Furthermore, every vertex is incident to exactly two edges and every edge is incident to exactly two vertices (see Figure 11). For this reason, a 2-polytope is called a *polygon*, or if it is finite and has p vertices (and hence also p edges), a *p-gon*. Finally, 3-polytopes are also often called *polyhedra*.

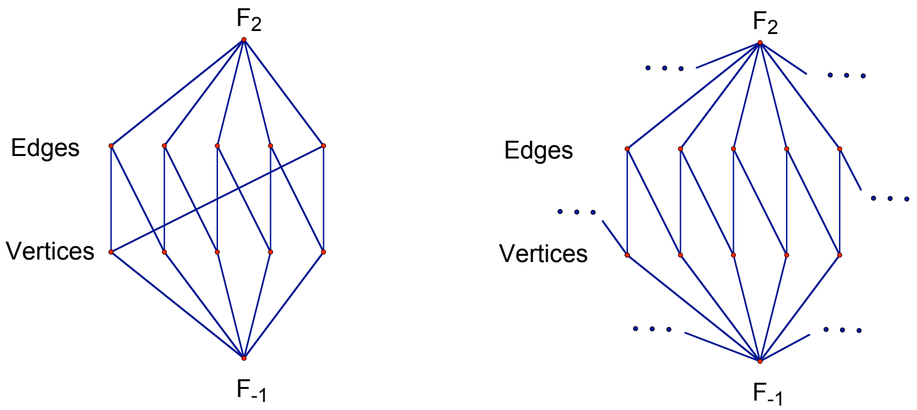

Figure 11: The (Hasse) diagram of a 5-gon and a ∞-gon, respectively.

Note that every two *geometric* p-gons are combinatorially equivalent. For example, a convex pentagon and a pentagram (in Figure 12) are different representations of the same abstract polytope, the 5-gon (in Figure 11).

Given two faces F and G of a polytope $\mathcal{P}$ such that $F \leq G$, the *section* G/F of $\mathcal{P}$ is the set of faces $\{H | F \leq H \leq G\}$. If F_0 is a vertex, then the section F_n/F_0 is called the *vertex-figure* of F_0. Note that every section G/F of a polytope $\mathcal{P}$ is also a polytope and has rank $\text{rank}(G/F) = \text{rank}(G) - \text{rank}(F) - 1$. Hence, the diamond condition states that all the sections of $\mathcal{P}$ of rank 1 have a diamond-shaped Hasse diagram.

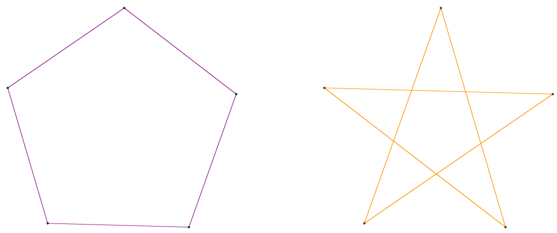

Figure 12: A convex pentagon and a pentagram are combinatorially equivalent.

We now reexamine the third condition in our definition of a polytope. A poset $\mathcal{P}$ is said to be *connected* if for any two proper faces (elements) F and G of $\mathcal{P}$, there is a sequence of proper faces $F = F^0, F^1, \ldots, F^k = G$ such that F^i and F^{i+1} are *incident* (i.e $F^i \leq F^{i+1}$ or $F^{i+1} \leq F^i$), for every $i = 0, \ldots, k-1$. A poset $\mathcal{P}$ is said to be *strongly connected* if every section of $\mathcal{P}$, including itself, is connected.

Proposition 37 *A poset $\mathcal{P}$ with a strictly monotone rank function having range $\{-1, \ldots, n\}$, a smallest (-1)-face F_{-1}, a greatest n-face F_n and such that each flag of $\mathcal{P}$ contains exactly $n+2$ faces is strongly connected if and only if it is strongly flag-connected.*

Let $\mathcal{P}$ be an n-polytope and Φ be a flag of $\mathcal{P}$. The diamond condition tells us that for $i = 0, \ldots, n-1$ there is exactly one flag that differs from Φ in the i-face. Such a flag is called the i-*adjacent* flag to Φ and it is denoted by Φ^i. Furthermore, we define $\Phi^{i,j} := (\Phi^i)^j$ and extend such notation by induction. We shall denote by $(\Phi)_i$ the i-face of the flag Φ. For convenience, we often omit the improper faces when describing a flag, thus, a flag Φ can be denoted as $\{(\Phi)_0, (\Phi)_1, \ldots, (\Phi)_{n-1}\}$. Two i-faces of $\mathcal{P}$, F and F', are said to be *adjacent* if there exists a flag Φ such that $(\Phi)_i = F$ and $(\Phi^i)_i = F'$.

Proposition 38 *For $i, j \in \{0, 1, ..., n-1\}$,*

1. $(\Phi^i)^i = \Phi$.
2. *if* $|i - j| > 1$, $\Phi^{i,j} = \Phi^{j,i}$.
3. $(\Phi)_i = (\Phi^j)_i$ *if and only if* $i \neq j$.

An n-polytope $\mathcal{P}$, $n \geq 2$, is said to be *equivelar* if, for each $j = 1, \ldots, n-1$, there exists an integer p_j, such that, for each flag $\Phi \in \mathcal{F}(\mathcal{P})$, the section $(\Phi)_{j+1}/(\Phi)_{j-2}$ is a p_j-gon. In this case, we say that $\mathcal{P}$ has *Schläfli type* (or sometimes only *type*) $\{p_1, p_2, \ldots p_{n-1}\}$. All 2-polytopes are equivelar; furthermore, a p-gon has Schläfli type $\{p\}$. We note that an infinite 2-polytope or *aperiogon* has Schläfli type $\{\infty\}$. A polyhedron is equivelar of Schläfli type $\{p, q\}$ if and only if all its facets are p-gons and all its vertex-figures are q-gons. In particular this implies that the Platonic Solids are equivelar. For example, a regular dodecahedron has type $\{5, 3\}$. Every two 2-polytopes with the same Schläfli type are isomorphic abstract polytopes. However this is not true for higher rank.

Exercise 39 *If we take a cube and identify opposite vertices, edges and faces, we obtain a* hemi-cube*. The hemi-cube can be represented in the projective plane as in Figure 13. What is the Schläfli type of the hemi-cube? Can you find an abstract polytopes that is not isomorphic to the hemi-cube, but has the same Schläfli type?*

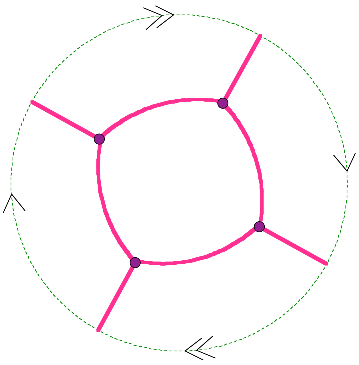

Figure 13: A hemi-cube.

An example of a 4-polytope is a hypercube or 4-cube. It has cubes as facets, three of them around each edge (see Figure 5) implying that its vertex-figures are tetrahedra. Hence, its Schläfli type is $\{4, 3, 3\}$.

Alternatively, the Schläfli type of an equivelar polytope can be defined as follows. For rank 2, we say that the Schläfli type of a p-gon is $\{p\}$. For higher rank, an $n-$polytope $\mathcal{P}$ is said to have Schläfli type $\{p_1, p_2, \ldots, p_{n-2}\}$ if all its facets have Schläfli type $\{p_1, p_2, \ldots, p_{n-1}\}$ and all its vertex-figures have Schläfli type $\{p_2, p_3, \ldots, p_{n-1}\}$.

Symmetries of polytopes

Let $\mathcal{P}$ and $\mathcal{Q}$ be two n-polytopes. An *isomorphism* from $\mathcal{P}$ to $\mathcal{Q}$ is a bijection $\gamma : \mathcal{P} \to \mathcal{Q}$ such that γ and γ^{-1} preserve the order. An *anti-isomorphism* $\delta : \mathcal{P} \to \mathcal{Q}$ is a bijection reversing the order, in which case $\mathcal{P}$ and $\mathcal{Q}$ are said to be *duals* of each other, and the usual convention is to denote $\mathcal{Q}$ by $\mathcal{P}^*$. (Note that $(\mathcal{P}^*)^* \cong \mathcal{P}$.) An isomorphism from $\mathcal{P}$ onto itself is called an *automorphism* of $\mathcal{P}$. An anti-isomorphism from $\mathcal{P}$ onto itself is called a *duality of* $\mathcal{P}$ (sometimes also *self-duality*). A polytope $\mathcal{P}$ is said to be *self-dual* if there exists a duality of $\mathcal{P}$. The set of all automorphisms and dualities of a polytope $\mathcal{P}$ forms a group, the *extended group* $\mathcal{D}(\mathcal{P})$ *of* $\mathcal{P}$, which contains $\mathrm{Aut}(\mathcal{P})$, the subgroup of all automorphisms of $\mathcal{P}$, as a subgroup of index at most 2. When a polytope $\mathcal{P}$ is not self-dual, then its extended group coincides with its automorphism group.

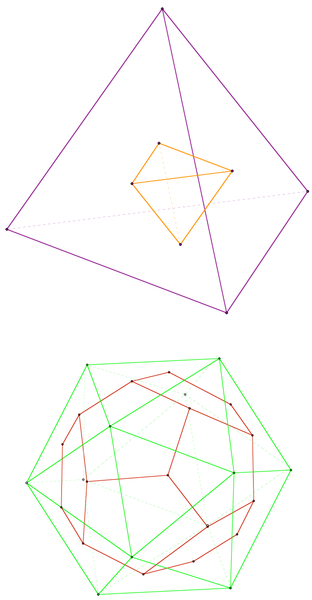

Figure 14: The tetrahedron is self-dual. The dodecahedron is dual to the icosahedron.

Lemma 40 *Let $\mathcal{P}$ be an n-polytope, let $\gamma \in \mathrm{Aut}(\mathcal{P})$ and let Φ be a flag of $\mathcal{P}$. Then,*

$$(\Phi^i)\gamma = (\Phi\gamma)^i \quad and \quad (\Phi)_i\gamma = (\Phi\gamma)_i,$$

for every $i = 0, 1, \ldots, n-1$.

Furthermore, if $\mathcal{P}$ is self-dual and δ is a duality of $\mathcal{P}$ then

$$(\Phi^i)\delta = (\Phi\delta)^{n-1-i} \quad and \quad (\Phi)_i\delta = (\Phi\delta)_{n-1-i},$$

for every $i = 0, 1, \ldots, n-1$.

Corollary 41 *Let $\mathcal{P}$ be an n-polytope, then $\mathrm{Aut}(\mathcal{P})$ acts freely on $\mathcal{F}(\mathcal{P})$, that is, every automorphism of $\mathcal{P}$ that fixes one flag is the identity.*

Lemma 42 *Let $\mathcal{P}$ be an n-polytope, and let us denote by $\mathrm{Orb}(\mathcal{P})$ the set of all flag orbits of $\mathcal{P}$ under the action of $\mathrm{Aut}(\mathcal{P})$. Let $\mathcal{O}_1, \mathcal{O}_2 \in \mathrm{Orb}(\mathcal{P})$ and $\Phi \in \mathcal{O}_1$. If for some $i \in \{0, \dots n-1\}$, $\Phi^i \in \mathcal{O}_2$, then for any $\Psi \in \mathcal{O}_1$, $\Psi^i \in \mathcal{O}_2$.*

Since $\mathrm{Aut}(\mathcal{P})$ acts freely on $\mathcal{F}(\mathcal{P})$, for each $\mathcal{O} \in \mathrm{Orb}(\mathcal{P})$ and each $\Phi \in \mathcal{O}$, there exists a bijection $\phi : \mathrm{Aut}(\mathcal{P}) \to \mathcal{O}$, $\phi : \gamma \mapsto \Phi\gamma$. Therefore there exists a bijection between every two flag orbits of $\mathcal{P}$. In particular, if $\mathcal{P}$ is finite, $|\mathrm{Aut}(\mathcal{P})| = |\mathcal{O}| = \frac{|\mathcal{F}(\mathcal{P})|}{|\mathrm{Orb}(\mathcal{P})|}$. Hence, $|\mathrm{Aut}(\mathcal{P})| \leq |\mathcal{F}(\mathcal{P})|$.

Regular polytopes

We started this course with a brief historical introduction to regular polytopes. We now formally define the concept of regularity. An (abstract) polytope is said to be *regular* if its automorphism group has exactly one orbit on the flags. Equivalently, a polytope $\mathcal{P}$ is regular if and only if $\mathrm{Aut}(\mathcal{P})$ is transitive on $\mathcal{F}(\mathcal{P})$.

Every regular convex polytope is a regular abstract polytope. The Kepler-Poinson polyhedra and the Petrie-Coxeter polyhedra are regular abstract polytopes.

Note that regular n-polytopes are equivelar since for each $j = 1, \dots, n-1$ and any two flags Φ, Ψ of the polytope, the sections $(\Phi)_{j+1}/(\Phi)_{j-2}$ and $(\Psi)_{j+1}/(\Psi)_{j-2}$ are isomorphic. Furthermore, for any $i, j \in \{0, \dots, n-1\}$ with $i < j$, the sections $(\Phi)_j/(\Phi)_i$ and $(\Psi)_j/(\Psi)_i$ are isomorphic and regular.

Exercise 43 *Is every equivelar n-polytope $\mathcal{P}$ a regular polytope?*

Exercise 44

1. *Show that every 2-polytope is regular.*
2. *Consider the following definition: Every 2-polytope is regular; a n-polytope ($n \geq 3$) is regular if its facets and vertex figures are regular and alike. Is this definition equivalent to our concept of regularity?*

Let $\mathcal{P}$ be an regular n-polytope of Schläfli type $\{p_1, p_2, \dots p_{n-1}\}$ and let Φ be a base flag of $\mathcal{P}$. Since $\mathrm{Aut}(\mathcal{P})$ is transitive on the flags, for every $i = 0, 1, \ldots, n-1$, there exists $\rho_i \in \mathrm{Aut}(\mathcal{P})$ such that $\Phi\rho_i = \Phi^i$.

Then,

$$\Phi\rho_i^2 = (\Phi^i)\rho_i = (\Phi\rho_i)^i = \Phi^{i,i} = \Phi;$$

also, if $|i-j| > 1$, then

$$\Phi\rho_i\rho_j = (\Phi^i)\rho_j = (\Phi\rho_j)^i = \Phi^{j,i} = \Phi^{i,j} = \Phi\rho_j\rho_i.$$

Since $\mathrm{Aut}(\mathcal{P})$ acts freely on $\mathcal{F}(\mathcal{P})$, this implies that

$$\rho_i^2 = (\rho_i\rho_j)^2 = \epsilon, \quad \text{if } |i-j| > 1. \qquad (1)$$

Lemma 45 *Suppose that $\mathcal{P}$ is a polytope such that for some base flag Φ there exist automorphisms ρ_i such that $\Phi\rho_i = \Phi^i$ for every $i \in \{0, \ldots, n-1\}$. Then $\mathcal{P}$ is a regular polytope. Furthermore, the automorphisms ρ_i are unique and they generate the automorphism group.*

We shall refer to $\rho_0, \rho_1, \ldots, \rho_{n-1}$ as the *distinguished generators* of $\mathcal{P}$ with respect to Φ.

For the reminder of this Section, unless otherwise stated we let $\mathcal{P}$ be a regular n-polytope, Φ be a base flag of $\mathcal{P}$ and let $\rho_0, \rho_1, \ldots, \rho_{n-1}$ be the distinguished generators of $\mathcal{P}$ with respect to Φ.

Lemma 46 *Let j and k such that $-1 \le j \le k \le n$, and consider the section $\mathcal{Q} := (\Phi)_k/(\Phi)_j$ of $\mathcal{P}$. Then $\mathcal{Q}$ is regular and*

$$\mathrm{Aut}(\mathcal{Q}) \cong \langle \rho_{j+1}, \rho_{j+2}, \ldots, \rho_{k-1} \rangle.$$

Corollary 47

1. *For each i, $0 < i < n$,*

$$\mathrm{Aut}((\Phi)_{i+1}/(\Phi)_{i-2}) \cong \langle \rho_{i-1}, \rho_i \rangle.$$

   Thus if $\{p_1, p_2, \ldots, p_{n-1}\}$ is the Schläfli type of $\mathcal{P}$, we have

$$(\rho_{i-1}\rho_i)^{p_i} = \epsilon.$$

2. *for any $J \subseteq \{0, \ldots, n-1\}$, let $\Phi_J := \{(\Phi)_j |\ j \in J\}$. Then the stabilizer (under the action of $\mathrm{Aut}(\mathcal{P})$) of Φ_J is precisely the group $\langle \rho_j |\ j \notin J \rangle$.*

Lemma 48 *Let $\mathcal{P}$ be a regular n-polytope, let Φ be a base flag of $\mathcal{P}$ and let $\rho_0, \rho_1, \ldots, \rho_{n-1}$ be the distinguished generators of $\mathcal{P}$ with respect to Φ. Then the distinguished generators of $\mathcal{P}$ satisfy the* intersection condition

$$\langle \rho_i |\ i \in I \rangle \cap \langle \rho_j |\ j \in J \rangle = \langle \rho_i |\ i \in I \cap J \rangle, \text{ for every } I, J \subseteq \{0, \ldots n-1\}.$$

A group that is a quotient of a sting Coxeter group and whose generators satisfy the intersection condition of lemma 48 is called a *string C-group.*

Theorem 49 *The automorphism group of a regular polytope is a string C-group. And conversely, every string C-group is the automorphism group of a regular polytope.*

Chiral polytopes

Coxeter defined a regular "map" (think 3-polytope) as a map whose automorphism group contains two particular elements: one that cyclically permutes consecutive edges in one face, and another which cyclically permutes the successive edges meeting at one vertex of this face. Note that the automorphism groups of such maps need not be transitive on the flags. When the automorphism group of a "regular" map possesses an automorphism ρ_0 which interchanges the two vertices of some edge without interchanging the incident two faces, the map becomes *reflexible* in Coxeter's terminology. If no such automorphism exists, then the map is *irreflexible.*

In more terms of abstract polytopes, reflexible maps are regular (abstract) polytopes while irreflexible maps are not (as their automorphism groups are not transitive on the flags). In fact, irreflexible maps are now often called chiral maps. Extending this idea to rank $n > 3$, we define a chiral n-polytope as follows.

An n-polytope $\mathcal{P}$ with base flag Φ is called *chiral* if it is not regular, but there exist automorphisms $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ such that each σ_i fixes all faces in Φ different from $(\Phi)_{i-1}$ and $(\Phi)_i$, and cyclically permutes consecutive i-faces of $\mathcal{P}$ in the rank 2 section $(\Phi)_{i+1}/(\Phi)_{i-2}$ of $\mathcal{P}$. Such automorphisms generate $\mathrm{Aut}(\mathcal{P})$ and are called the *distinguished generators* of $\mathrm{Aut}(\mathcal{P})$ with respect to Φ.

Let Ψ be a flag of a polytope $\mathcal{P}$. We say that Ψ is *even with respect to* Φ if there exists a sequence of adjacent flags $\Phi = \Phi_0, \Phi_1, \ldots \Phi_{2k-1}, \Phi_{2k} = \Psi$. If $\Psi \in \mathcal{F}(\mathcal{P})$ is not even, then we say that it is *odd (with respect to* Φ*).* It is not hard to see that the orbit of Φ under the automorphism group of a chiral polytope $\mathcal{P}$ is precisely the set of all even flags with respect to Φ. This implies that odd flags exist in $\mathcal{P}$ and thus, the automorphism group of a chiral polytope has two orbits on the flags (the set of even flags and the set of odd flags). Furthermore, all the flags adjacent to an even flag are odd (and all flags adjacent to an odd flag are even). Hence, chiral polytopes are precisely those polytopes whose automorphism group has exactly two orbits on the flags, with adjacent flags in different orbits.

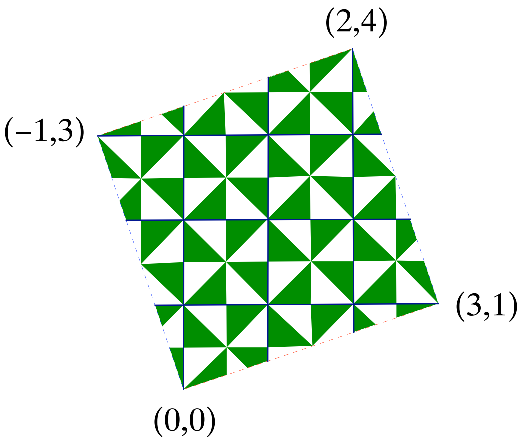

Figure 15: A chiral polytope on the torus.

Chiral polytopes can be said to occur in pairs of *enantiomorphic forms*, with one being the 'mirror image' of the other (see Figure 16 for an example)

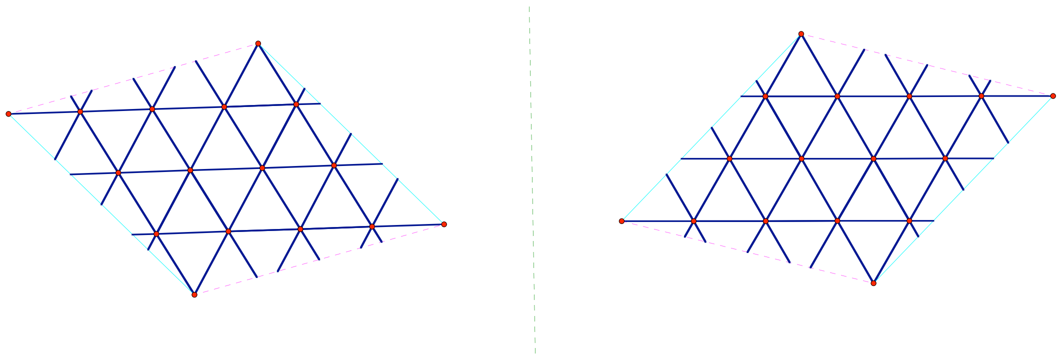

Figure 16: A chiral polyhedron and its enantiomorphic form.