Sobolev Spaces and Newton-Lagrange interpolation

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Abstract

In the talk I will discuss the proof and some applications of the following Theorem. A measurable function $f(x)$ in $\mathbb{R}^n$ (or its subdomain) is in the Sobolev class $W^{m,p}(\mathbb{R}^n), p > 1$, iff the pointwise inequality

$\nabla^m f$

$$\Delta^m f(x;y) \equiv \left|\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j} f(x+jh)\right| \leq |x-y|^m[a_f(x)+a_f(y)] \tag{0.1}$$

holds for some function $a_f^m(x)$ in the Lebesgue class $L^p(\mathbb{R}^n)$ for any two points x, y in $\mathbb{R}^n$ (off a set of measure zero). ~~Here~~ $\cancel{h = \frac{y-x}{m}}$.

Analogues of inequality (0.1) also hold for arbitrary interpolating nodes in the segment $[x, y]$. Then, instead of finite differences $\Delta^m f(x;y)$, the divided differences $f[x_0, \ldots, x_m], x_0 = x, \ldots, x_m = y$ come into play. Inequalities (0.1) directly relate the Sobolev classes with their numerical treatment, essential in all applications in mathematical modeling and data processing.

Theorem above seems to have been overlooked even in the most classical case for real valued functions on the real line ($n = 1$). The case $m = 1$ served as a basic concept in the development of Sobolev spaces theory and harmonic analysis on abstract measure metric spaces.