Sobolev Spaces and the Whitney Extension Theorem

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Sobolev spaces

$$W^{m,p}(\Omega) = \{ u \in L^p(\Omega) \,/\, D^\alpha u \in L^p, \ \ |\alpha| \leq m \}$$

$$\|u\|_{m,p} = \sum_{|\alpha| \leq m} \|D^\alpha u\|_p .$$

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No fractional order spaces.

Theorem (Calderón-Zygmund 1961)

Let $u \in W^{m,p}(\Omega)$, $\Omega \subset \mathbb{R}^n$. Then for any $\varepsilon > 0$ there is $g \in C^m(\Omega)$ such that

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Theorem (Michael-Ziemer 1985)

Let $u \in W^{m,p}(\Omega)$, $\Omega \subset \mathbb{R}^n$, $1 < p < \infty$, $1 \leq k \leq m$. Then for any $\varepsilon > 0$ there is $g \in C^k(\Omega)$ such that

$$B_{m-k,p}(\{u \neq g\}) < \varepsilon$$

$$\|u - g\|_{k,p} < \varepsilon.$$

Bojarski - H. 1993 - simplified proof.

All the proofs are based on the Whitney extension theorem. The proof due to Bojarski - H. is based on pointwise inequalities and it goes as follows.

Taylor polynomial

$$T_x^{\ell} u(y) = \sum_{|\alpha| \leq \ell} D^{\alpha} u(x) \frac{(y-x)^{\alpha}}{\alpha !}$$

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Maximal functions

$$M_R f(x) = \sup_{r<R} \fint_{B(x,r)} |f(y)| \, dy$$

$$M_R^{b} f(x) = \sup_{r<R} \fint_{B(x,r)} |f(y) - f(x)| \, dy$$

For $u \in W^{m,p}(\mathbb{R}^n)$ we have pointwise inequalities:

$$\frac{|u(y) - T_x^{m-1} u(y)|}{|x-y|^m} \le C \left(M_{|x-y|} |\nabla^m u|(x) + M_{|x-y|} |\nabla^m u|(y) \right)$$

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$$\frac{|u(y)-T_x^{m}u(y)|}{|x-y|^m} \le C\left(M^b_{|x-y|}\nabla^m u(x) + M^b_{|x-y|}\nabla^m u(y)\right)$$

Applying it to $D^\alpha u$, $|\alpha| \le m$ we obtain

$$\frac{|D^\alpha u(y) - T^{m-|\alpha|} D^\alpha u(y)|}{|x-y|^{m-|\alpha|}} \le C \left(M^\flat_{|x-y|} \nabla^m u(x) + M^\flat_{|x-y|} \nabla^m u(y) \right)$$

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If on a compact set K

$$M^\flat_R \nabla^m u \rightrightarrows 0 \quad \text{as} \quad R \to 0$$

it follows from the Whitney extension theorem that

$$u_{|K} \text{ extends to } g \in C^m(\mathbb{R}^n).$$

This proves the Calderón-Zygmund-Liu theorem.

If $k \leq m$

$$\frac{|D^\alpha u(y) - T^{k-|\alpha|} D^\alpha u(y)|}{|x-y|^{k-|\alpha|}} \leq C\left(M^b_{|x-y|} \nabla^k u(x) + M^b_{|x-y|} \nabla^k u(y)\right)$$

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and

$$M^b_R \nabla^k u \rightrightarrows 0 \quad \text{as} \quad R \to 0$$

on a closed set whose complement has small $B_{m-k,p}$ capacity. This proves

Theorem (Michael-Ziemer 1985)

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Then for any $\varepsilon > 0$ there is a function $g \in C^k(\Omega)$ such that

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Theorem (Bojarski–H.–Strzelecki 2002)

Let $u \in W^{m,p}(\Omega)$, $\Omega \subset \mathbb{R}^n$, $1 < p < \infty$, $0 \leq k < m$. Then for any $\varepsilon > 0$ there is $g \in C^k(\Omega) \cap W^{k+1,p}$ such that

$$B_{m-k,p}(\{u \neq g\}) < \varepsilon$$

$$\|u - g\|_{k+1,p} < \varepsilon.$$

Using the Whitney extension from K we lose information about $u/_{\Omega \setminus K}$. In order to retain this information we can use the <u>Whitney smoothing</u>

$\mathbb{R}^n \setminus K = \bigcup_i Q_i$ Whitney's decomposition

$\{\varphi_i\}$ partition of unity

$$g(x) = \begin{cases} u(x) & \text{if } x \in K \\ \sum_i \varphi_i(x) \fint_{2Q_i} (T_z^k u(x))\, dz & \text{if } x \in K^c \end{cases}$$

This argument was used in the proof of the Bojarski-H.-Strzelecki theorem.

Functions of bounded variation

$$BV(\Omega) = \{ u \in L^1 \mid Du \text{ measure of finite total variation} \}$$

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<u>Theorem</u> If $u \in BV(\Omega)$, then for any $\varepsilon > 0$ there is $g \in C^\infty(\Omega)$ such that

$$\|u - g\|_1 < \varepsilon$$

$$\big|\, \|Du\|(\Omega) - \|Dg\|(\Omega) \big| < \varepsilon.$$

Higher order spaces

$$BV^m(\Omega) = \left\{ u \in L^1 \,\middle|\, D^\alpha u,\ |\alpha| \leq m \ \text{ measures of finite total variation} \right\}$$

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Theorem If $u \in BV^m(\Omega)$, then for any $\varepsilon > 0$ there is $g \in C^\infty(\Omega)$ such that

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Alberti – applications to the investigation of structure of singularities of convex functions.

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Francos 2011 - detailed proof based on pointwise inequalities. From his proof it follows that

$$\|u - g\|_{m-1,1} < \varepsilon.$$

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Recall the result:

Theorem If $u \in BV^m(\Omega)$, then for any $\varepsilon > 0$ there is $g \in C^\infty(\Omega)$ such that

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Some progress using Whitney's smoothing (Francos, H., Korobkov), but the answer is stil unclear.

Theorem (Bourgain, Korobkov, Kristensen 2012)

Let $u \in BV^m(\mathbb{R}^n)$, $2 \leq m \leq n$. Then for any $\varepsilon > 0$ there is $g \in C^{m-2,1}$ such that

$$\mathcal{H}^{n-1}_{\infty}(\{u \neq g\}) < \varepsilon$$

$$\| u - g \|_{BV^m} < \varepsilon .$$

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The proof follows a variant of the method of Whitney's smoothing from Bojarski - H. - Strzelecki 2002.

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Applications to the Sard theorem for BV^m mappings.

Theorem (Sard 1942)

$f \in C^k(\mathbb{R}^m, \mathbb{R}^n)$, $k > \max(m-n, 0) \Rightarrow |f(\operatorname{Crit} f)| = 0$.

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The Sard theorem is not true for

$f \in C^k$, $k \leq m-n$.

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Sobolev mappings $W^{k,p}$ are C^k smooth on large sets, so a version of the Sard theorem should be true for Sobolev mappings.

Problem:

When dealing with Sobolev mappings $W^{k,p}$ we rarely assume k to be large and hence the smoothness k might be too low for the Sard theorem to hold.

Is there a version of the Sard theorem for $k \leq m-n$?

Theorem (Dubovitskiĭ 1957)

If $f \in C^k(\mathbb{R}^m, \mathbb{R}^n)$, $s = m-n-k+1$, then

$$\mathcal{H}^s(f^{-1}(y) \cap \mathrm{Crit}\, f) = 0 \text{ for a.e. } y \in \mathbb{R}^n.$$

$$f^{-1}(y) = \underbrace{(f^{-1}(y) \setminus \mathrm{Crit}\, f)}_{(m-n) \text{ manifold}} \cup \underbrace{(f^{-1}(y) \cap \mathrm{Crit}\, f)}_{\text{small Hausdorff dimension}}$$

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$k > \max(m-n, 0) \Rightarrow s \leq 0$, $\mathcal{H}^s$ – counting measure

$$\mathcal{H}^s(f^{-1}(y) \cap \mathrm{Crit}\, f) = 0 \text{ for a.e. } y \iff$$

$$f^{-1}(y) \cap \mathrm{Crit}\, f = \emptyset \text{ for a.e. } y \iff$$

$$|f(\mathrm{Crit}\, f)| = 0$$

Thus

Dubovitskiĭ $\Rightarrow$ Sard.

Bojarski - H. - Strzelecki 2005 - new proof of the Dubovitskiĭ theorem and some generalizations.

The Dubovitskiĭ and the Calderón-Zygmund theorems imply

Theorem (Bojarski-H.-Strzelecki 2005)

Let $f \in W^{k,p}(\mathbb{R}^m, \mathbb{R}^n)$. Then there is a Borel representative of f such that for a.e. $y \in \mathbb{R}^n$ we have

$$f^{-1}(y) = Z \cup \bigcup_{j=1}^{\infty} K_j$$

where

$$\mathcal{H}^{m-n-k+1}(Z) = 0$$

and

$$K_j \subset K_{j+1}, \quad K_j \subset S_j \qquad j = 1, 2, 3, \ldots$$

S_j is an $(m-n)$-submanifold of $\mathbb{R}^m$.

Other versions of the Sard theorem in the Sobolev setting:

Theorem (De Pascale 2001)

Let $f \in W^{k,p}(\mathbb{R}^m, \mathbb{R}^n)$, $m > n$, $p > m$, $k > m-n$. Then $|f(\mathrm{Crit}\, f)| = 0$.

Under the given assumptions $f \in C^1$, so $\mathrm{Crit}\, f$ is defined in the classical way.

Theorem (Bourgain–Korobkov–Kristensen 2012)

If $f \in W^{n,1}(R^n)$, then for a.e. $y \in R$, $f^{-1}(y)$ is a finite disjoint family of $(n-1)$-dimensional C^1 manifolds without boundary.

They also have a version of the Sard theorem for $BV^n(R^n)$ functions.

Sobolev extensions

$\mathcal{P}^m$ - polynomials on $\mathbb{R}^n$, degree $\leq m$

$$\mathcal{E}_m(f; x, r) = \inf_{P \in \mathcal{P}^{m-1}} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f - P|$$

$$f_m^{\#}(x) = \sup_{r>0} r^{-m} \mathcal{E}_m(f; x, r)$$

Theorem (Calderón 1972)

$f \in W^{m,p}(\mathbb{R}^n)$, $1 < p \leq \infty$ iff

$f \in L^p$ and $f_m^{\#} \in L^p$

$$\|f\|_{m,p} \approx \|f\|_p + \|f_m^{\#}\|_p$$

$E \subset \mathbb{R}^n$ positive measure

$$\mathcal{E}_{m,E}(f;x,r) = \inf_{P \in \mathcal{P}^{m-1}} \frac{1}{|B(x,r)|} \int_{B(x,r)\cap E} |f-P|$$

$$f^{\#}_{m,E}(x) = \sup_{r>0} r^{-m}\, \mathcal{E}_{m,E}(f;x,r)$$

<u>Calderón space</u>

$$C^{m,p}(E) = \{ f \in L^p(E) \mid f^{\#}_{m,E} \in L^p(E) \}$$

$$\|f\|_{C^{m,p}(E)} = \|f\|_{p,E} + \|f^{\#}_{m,E}\|_{p,E}$$

Calderón's result states that

$$W^{m,p}(\mathbb{R}^n) = C^{m,p}(\mathbb{R}^n).$$

$E \subset \mathbb{R}^n$ satisfies the <u>measure density condition</u> if

$$|E \cap B(x,r)| \geq c\, r^n, \quad x \in E, \quad 0 < r \leq 1.$$

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Rychkov 2000 (special case)
Shvartsman 2006 (general case) proved

Theorem If $E \subset \mathbb{R}^n$ satisfies $(*)$ and $1 < p < \infty$, then

$$W^{m,p}(\mathbb{R}^n)|_E = C^{m,p}(E)$$

with equivalent norms. Moreover there is a bounded extension operator

$$E : C^{m,p}(E) \rightarrow W^{m,p}(\mathbb{R}^n)$$

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$$\mathcal{E} : C^{m,p}(E) \to W^{m,p}(\mathbb{R}^n)$$

$\mathcal{E}$ is constructed explicitely. It is a variant of the Whitney - Jones extension.

Theorem (H.–Koskela–Tuominen 2008)

$\Omega \subset \mathbb{R}^n$ arbitrary domain, $1<p<\infty$, $m \geq 1$.
The following conditions are equivalent:

(1) For every $f \in W^{m,p}(\Omega)$ there is $F \in W^{m,p}(\mathbb{R}^n)$ such that $F|_\Omega = f$.

(2) The trace operator
$$Tr : W^{m,p}(\mathbb{R}^n) \to W^{m,p}(\Omega)$$
is surjective

(3) There is a bounded extension operator
$$E : W^{m,p}(\Omega) \to W^{m,p}(\mathbb{R}^n).$$

(4) Ω satisfies the measure density condition and $W^{m,p}(\Omega) = C^{m,p}(\Omega)$.

Corollary (H.-Koskela-Tuominen 2008)

Let $\Omega, G \subset \mathbb{R}^n$ be two domains that are bi-Lipschitz homeomorphic. Then Ω is a $W^{1,p}$-extension domain for some $1 < p \leq \infty$ iff G is a $W^{1,p}$-extension domain.

Theorem (Koskela 1998)

If $W^{1,p}(\Omega) \subset C^{1-\frac{n}{p}}(\overline{\Omega})$ for some $p>n$, then there is a bounded extension operator

$$E : W^{1,q}(\Omega) \to W^{1,q}(\mathbb{R}^n)$$

for all $q>p$.

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Theorem (Gong-H.-Koskela-Zhong 2012)

Let $n < p < \infty$. The following conditions are equivalent

(1) $W^{1,p}(\Omega) \subset C^{1-\frac{n}{p}}(\overline{\Omega})$,

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(3) Ω is a $W^{m,p}$-extension domain for all $m \geq 1$.

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If Ω is a finitely connected bounded planar domain, equivalence of (1) & (2) has been proved by Shvartsman 2010

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(2) Ω is a $W^{1,p}$-extension domain,

(3) Ω is a $W^{m,p}$-extension domain for all $m \geq 1$.

The proof requires techniques from the analysis on metric spaces even though the statement refers only to Euclidean spaces.

Corollary (Gong-H.-Koskela-Zhong 2012)

If Ω is a $W^{1,p}$-extension domain for some $p \geq n$, then it is a $W^{m,q}$-extension domain for all m and $q > p$.