Unitarizable representations and amenable operator algebras

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[Minor corrections made to slides after talk]

Today's menu

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Let Γ be a discrete group and A be a unital C*-algebra. We say a bounded representation $\theta : \Gamma \to A_{\text{inv}}$ is unitarizable, or similar to a unitary representation, if there exists some $s \in A_{\text{inv}}$ such that

$$s\theta(x)s^{-1} \in \mathcal{U}(A) \quad \text{for all } x \in \Gamma.$$

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Let A_{inv}^{+} be the subset of **positive** invertible elements. Then Γ acts on A_{inv}^{+} by

$$\theta^{+}(x) : h \mapsto \theta(x)h\theta(x)^{*}$$

Exercise

Show that $\theta : \Gamma \to A_{\mathrm{inv}}$ is unitarizable if and only if the action of Γ on A_{inv}^{+} has a fixed point.

Example 1. Let $e \in A$ be an idempotent. Define $\theta : \mathbb{Z}/2\mathbb{Z} \to A$ by $\theta(1) = 2e - 1_A$. Then θ is a bounded unitarizable representation; equivalently, e is similar to a projection.

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Example 2. Let $\varepsilon > 0$ and consider

$$a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad , \quad b = \begin{pmatrix} 1 & \varepsilon \\ 0 & 0 \end{pmatrix} .$$

These correspond to involutions $x = 2a - I_2$ and $y = 2b - I_2$ in $\mathbb{M}_2$, which give a pair of representations $\theta_x, \theta_y : \mathbb{Z}/2\mathbb{Z} \to (\mathbb{M}_2)_{\text{inv}}$.

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The following result is, essentially, due to Day (1950) and Dixmier (1950). It unifies earlier results of Lorch and Sz.-Nagy.

Theorem

Let Γ be an amenable discrete group and $\mathcal{M}$ a von Neumann algebra. Then every bounded representation $\Gamma \to \mathcal{M}$ is unitarizable.

Remark

This unitarizability property characterizes those discrete groups which are amenable, provided one tests over all von Neumann algebras. (Pisier, 2007)

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Some evidence in favour of this:

Theorem (Brannan–Samei, 2010)

Let G be a SIN group. Then every completely bounded homomorphism $\mathrm{A}(G) \to \mathcal{B}(\mathsf{H})$ is similar to a $*$-homomorphism.

Remark: if G has a closed copy of $\mathbf{F}_2$ then the adverb "completely" cannot be removed (C.–Samei, 2013)

For other results in the LCQG setting, see Brannan–Daws–Samei (2013).

Today's menu

Quick definition: a Banach algebra $\mathfrak{A}$ is amenable if it has a bounded approximate diagonal, i.e. a bounded net $(m_\alpha) \in \mathfrak{A} \widehat{\otimes} \mathfrak{A}$ satisfying $a \cdot m_\alpha - m_\alpha \cdot a \to 0$ and $a\pi(m_\alpha) \to a$ for each $a \in \mathfrak{A}$.

Example 3. [Johnson, 1972] If Γ is a discrete amenable group, then $\ell^1(\Gamma)$ is amenable.

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Amenability has good hereditary properties, for example:

- if $\mathfrak{A}$ is amenable and $\theta : \mathfrak{A} \to \mathfrak{B}$ is a homomorphism with dense range, then $\mathfrak{B}$ is amenable;
- if $\mathfrak{A}$ is a Banach algebra, $\mathfrak{J}$ is a closed ideal in $\mathfrak{A}$, and $\mathfrak{J}$ and $\mathfrak{A}/\mathfrak{J}$ are both amenable, then so is $\mathfrak{A}$.

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Example 5. [Rosenberg, 1977] The algebras $\mathcal{O}_n$, $2 \leq n \leq \infty$, are amenable (but not strongly amenable).

Example 6. [Bunce, 1976] Let Γ be a **discrete** non-amenable group. Then $C_r^*(\Gamma)$ is not amenable.

Remark

None of the proofs of these results ever need to mention the word "nuclear".

Every amenable, finite-dimensional algebra is (isomorphic to) a direct sum of full matrix algebras (WEDDERBURN's theorem).

So if $\mathfrak{A} \subseteq \mathbb{M}_n$ is an amenable subalgebra, then it is isomorphic **as a Banach algebra** to a C^*-algebra. (However it need not be a self-adjoint subalgebra of $\mathbb{M}_n$.)

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Question.

Let $\mathfrak{A}$ be a closed subalgebra of $\mathcal{B}(\mathsf{H})$. If $\mathfrak{A}$ is amenable, must it be isomorphic to a C*-algebra?

Let $\mathcal{Q}(\mathsf{H}) := \mathcal{B}(\mathsf{H})/\mathcal{K}(\mathsf{H})$ be the Calkin algebra and $q : \mathcal{B}(\mathsf{H}) \to \mathcal{Q}(\mathsf{H})$ the quotient homomorphism.

Question.

Is every bounded representation $\mathbb{Z} \to \mathcal{Q}(\mathsf{H})$ unitarizable? What if we replace $\mathbb{Z}$ by some other discrete abelian group?

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The point of Ozawa's question: if Γ is abelian then

- each bounded rep $\theta : \Gamma \to \mathcal{Q}(\mathsf{H})$ gives an amenable $\mathfrak{A} \subset \mathcal{B}(\mathsf{H})$;
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The contrapositive

If $\theta : \Gamma \to \mathcal{Q}(\mathsf{H})$ is bounded and **non-unitarizable**, then $\mathfrak{A}$ will be amenable yet not isomorphic to any C*-algebra.

Details

Given $\theta : \Gamma \to \mathcal{Q}(\mathsf{H})$ define $\mathfrak{B} = \overline{\text{lin}}\{\theta(x) : x \in \Gamma\}$.
$\mathfrak{B}$ is amenable (since it contains $\ell^1(\Gamma)$ is a dense subalgebra).

Let $\mathfrak{A} = q^{-1}(\mathfrak{B})$. There is a short exact sequence

$$0 \to \mathcal{K}(\mathsf{H}) \to \mathfrak{A} \xrightarrow{q} \mathfrak{B} \to 0$$

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Now suppose $\mathfrak{A}$ is also isomorphic to a C*-algebra. Then there exists $R \in \mathcal{B}(\mathsf{H})_{\text{inv}}$ such that $R\mathfrak{A}R^{-1}$ is a **self-adjoint subalgebra** of $\mathcal{B}(\mathsf{H})$.

Put $s := q(R)$. Then $s\mathfrak{B}s^{-1}$ is a commutative and self-adjoint subalgebra of $\mathcal{Q}(\mathsf{H})$. Observe: if $x \in \Gamma$, then $s\theta(x)s^{-1}$ is normal with spectrum contained in $\mathbb{T}$, hence is unitary. So s unitarizes θ.

Today's menu

1. Unitarizable representations
2. Amenable operator algebras?
3. Non-unitarizable representations
4. Unitarizable representations (reprise)
5. Questions

Theorem (see arXiv:1309.2415v1)

There is a set $\mathfrak{T}$ of pairwise distinct, bounded representations $\bigoplus_{\mathfrak{c}} \mathbb{Z} \to \mathcal{Q}(\ell_2)$, with $|\mathfrak{T}| = 2^{\mathfrak{c}}$, such that

- *$\mathfrak{T}$ is parametrized by certain "1-cocycles" $\bigoplus_{\mathfrak{c}} \mathbb{Z} \to \mathcal{Q}(\ell_2)$*
- *$\theta \in \mathfrak{T}$ is unitarizable iff it corresponds to an* **"inner" cocycle**.

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But inner cocycles are parametrized by elements of $\mathcal{Q}(\ell_2)$, and $|\mathcal{Q}(\ell_2)| = \mathfrak{c} < 2^{\mathfrak{c}} = |\mathfrak{T}|$. Therefore:

Corollary (Farah, Ozawa, ibid.)

There exists a non-unitarizable representation $\theta : \bigoplus_{\mathfrak{c}} \mathbb{Z} \to \mathcal{Q}(\ell_2)$. Hence, by our previous discussions, there exists an amenable closed subalgebra $\mathfrak{A} \subset \mathcal{B}(\ell_2)$ that is not isomorphic to any C-algebra.*

Write $\bigoplus_{\mathfrak{c}} \mathbb{Z} = \varinjlim_X \bigoplus_X \mathbb{Z}$ where the inductive limit is over all countable subsets $X \subset \mathfrak{c}$. It turns out that the restriction of θ to each $\mathbb{Z}^X$ **is** unitarizable, and that similarity elements can be chosen in a uniformly bounded way.

This gives the algebra $\mathfrak{A}$ another striking feature: we have $\mathfrak{A} = \lim_X \mathfrak{A}_X$ where each $\mathfrak{A}_X$ is separable and similar to a C*-algebra, and similarity elements s_X exist with $\sup_X \|s_X\| < \infty$.

Question.

Can we interpret this in terms of algebra-valued sheaves on the Pontryagin dual of $\bigoplus_{\mathfrak{c}} \mathbb{Z}$?

The previous construction uses a family of $\mathfrak{c}$-many pairwise-orthogonal projections in $\mathcal{Q}(\ell_2)$. In fact the family lives in ℓ_∞/c_0.

Moreover, we can make do with "only $\aleph_1$-many" projections, and replace an abstract counting argument with an inductive construction.

Also, it turns out that the rank of the group, not the absence of torsion, is the key. This also allows one to replace the cocycle machinery with explicit 2×2 matrix arguments.

The upshot: we can construct subhomogeneous examples!

Note: any amenable closed subalgebra of ℓ^∞ is isomorphic to some $C(X)$ (Sheinberg, 1977)

Theorem (C.–Farah–Ozawa, 2014)

There is a non-unitarizable representation $\bigoplus_{\aleph_1} \mathbb{Z}/2\mathbb{Z} \to (\ell^\infty/c_0) \otimes \mathbb{M}_2$. *This gives rise to an amenable subalgebra of* $\ell^\infty \otimes \mathbb{M}_2$ *which has density character* $\aleph_1$ *and is not isomorphic to any* C^**-algebra.*

The algebra $\mathfrak{A}$ is, as before, the inductive limit of separable algebras which are similar to C^* with uniform bounds on the similarities. For any $K > 1$, we can arrange that $\mathfrak{A}$ has a bounded approximate diagonal of norm $\leq K$.

Further sharpened by Vignati, `arXiv 1402.1112`, to get an version with the **additional property** that none of its nonseparable amenable subalgebras are isomorphic to C^*-algebras.

Let $\Gamma = \bigoplus_{\aleph_1} \mathbb{Z}/2\mathbb{Z}$. The trick is to find two commuting, bounded representations $\theta_x, \theta_y : \Gamma \to (\ell^\infty/c_0) \otimes \mathbb{M}_2$ which cannot be simultaneously unitarized.

Then $\theta_x \times \theta_y : \Gamma \times \Gamma \to (\ell^\infty/c_0) \otimes \mathbb{M}_2$ is the desired bounded but non-unitarizable representation.

All the real work takes place inside (ℓ_∞/c_0).

We can find $\mathcal{F}, \mathcal{G} \subset 2^{\mathbb{N}}$, with $|\mathcal{F}| = |\mathcal{G}| = \aleph_1$, such that

$$(q(1_{\mathsf{J}}))_{\mathsf{J}\in\mathcal{F}} \cup (q(1_{\mathsf{K}}))_{\mathsf{K}\in\mathcal{G}}$$

is a family of non-zero, pairwise-orthogonal projections in ℓ_∞/c_0.

We can also arrange for the following condition to hold.

"Magic condition"

For each $\mathsf{X} \subset \mathbb{N}$, either there exists $\mathsf{J} \in \mathcal{F}$ such that $q(1_{\mathsf{X}})q(1_{\mathsf{J}}) \neq 0$, or there exists $\mathsf{K} \in \mathcal{G}$ such that $q(1_{\mathbb{N}\setminus\mathsf{X}})q(1_{\mathsf{K}}) \neq 0$.

Now pick two involutions $x, y \in \mathbb{M}_2$ for which the actions on $(\mathbb{M}_2)^+_{\text{inv}}$ have **no common fixed point.**

For each $\mathsf{J} \in \mathcal{F}$ and $\mathsf{K} \in \mathcal{G}$ we can define involutions in $\ell_\infty \otimes \mathbb{M}_2$:

$$x_{\mathsf{J}} = 1_{\mathsf{J}} \otimes x + 1_{\mathbb{N}\setminus\mathsf{J}} \otimes I_2 \quad \text{and} \quad y_{\mathsf{K}} = 1_{\mathsf{K}} \otimes y + 1_{\mathbb{N}\setminus\mathsf{K}} \otimes I_2.$$

Define $\theta_x : \Gamma \to (\ell_\infty / c_0) \otimes \mathbb{M}_2$ by

$$\theta_x(e_{\mathsf{J}}) = (q \otimes \mathrm{id})(x_{\mathsf{J}}) \qquad (\mathsf{J} \in \mathcal{F}),$$

and define θ_y similarly. These representations of Γ are bounded and their ranges commute.

Suppose θ_x^+ and θ_y^+ have a **common fixed point**, say $q(s)$ for some positive invertible $s = (s_n) \in \ell_\infty \otimes \mathbb{M}_2$. We can show that this contradicts the "magic condition" on our families $\mathcal{F}$ and $\mathcal{G}$.

If such $s = (s_n)$ exists then

$$(xs_nx^* - s_n)_{n\in\mathsf{J}} \in c_0(\mathsf{J}) \otimes \mathbb{M}_2 \quad \text{for all } \mathsf{J} \in \mathcal{F}$$

$$(ys_ny^* - s_n)_{n\in\mathsf{K}} \in c_0(\mathsf{K}) \otimes \mathbb{M}_2 \quad \text{for all } \mathsf{K} \in \mathcal{G}$$

Using $\sup_n \|s_n\| < \infty$ and $\sup_n \left\|s_n^{-1}\right\| < \infty$, some work yields

- $\inf_n \operatorname{dist}(s_n, \operatorname{Fix}_x) + \operatorname{dist}(s_n, \operatorname{Fix}_y) = \delta > 0$;
- $(\operatorname{dist}(s_n, \operatorname{Fix}_x))_{n\in\mathsf{J}} \in c_0(\mathsf{J})$ for all $\mathsf{J} \in \mathcal{F}$;
- $(\operatorname{dist}(s_n, \operatorname{Fix}_y))_{n\in\mathsf{K}} \in c_0(\mathsf{K})$ for all $\mathsf{K} \in \mathcal{G}$.

From these constraints we deduce: there are subsets $\mathsf{X}, \mathsf{Y} \subseteq \mathbb{N}$, with $\mathsf{X} \cup \mathsf{Y} = \mathbb{N}$, and $|\mathsf{X} \cap \mathsf{J}| < \infty$ for all $\mathsf{J} \in \mathcal{F}$, and $|\mathsf{Y} \cap \mathsf{K}| < \infty$ for all $\mathsf{K} \in \mathcal{G}$. **This contradicts the magic condition**, as required. □

Can we use the same machine to create separable "counter-examples", using some complicated but countable amenable group Γ instead of big abelian groups?

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Theorem (C.–Farah–Ozawa, *ibid.*)

Let Γ be a **countable** *amenable group. Then every bounded representation $\Gamma \to \mathcal{Q}(\mathsf{H})$ is unitarizable.*

The same is true if we replace the Calkin algebra by other kinds of corona algebra such as $\prod_n \mathbb{M}_n / \bigoplus \mathbb{M}_n$ or $(\ell^\infty / c_0) \otimes \mathbb{M}_n$, or ultraproducts of a sequence of C*-algebras.

Let Γ be a discrete group, A a unital C*-algebra, $\theta : \Gamma \to A_{\text{inv}}$ a bounded representation.

If $h \in A_{\text{inv}}^{+}$ and $\theta(x)h\theta(x)^* = h$ for all $x \in \Gamma$, then

$$h^{-1/2}\theta(x)h^{1/2} \in \mathcal{U}(A) \quad \text{for all } x \in \Gamma.$$

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If $h \in A^+_{\text{inv}}$ and $\theta(x)h\theta(x)^* = h$ for all $x \in \Gamma$, then

$$h^{-1/2}\theta(x)h^{1/2} \in \mathcal{U}(A) \quad \text{for all } x \in \Gamma.$$

A standard theme: when looking for a fixed point of a (semi)group action, try to take an "average over an orbit".

So now suppose Γ has a Følner **sequence** (F_n).

Put $h_n = \frac{1}{|F_n|} \sum_{y \in F_n} \theta(y)\theta(y)^*$. Then for any $x \in \Gamma$,

$$\|\theta(x) h_n \theta(x)^* - h\| \le |F_n|^{-1} |xF_n \triangle F_n| \, \|\theta\|^2 \to 0\,,$$

so (h_n) is an "asymptotically invariant" sequence in A_{inv}^+.

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so (h_n) is an "asymptotically invariant" sequence in A^+_{inv}.

The key point

If A has a certain "countable saturation property", tools from the metric model theory of C*-algebras allow us to construct the desired h from the sequence (h_n).

(These tools are an axiomatic version of ideas used by Pedersen to studying derivations from separable C*-algebras into corona algebras.)

Theorem (C., 2013)

Let A be a closed, commutative subalgebra of a **finite** *von Neumann algebra. If A is (operator) amenable, then A is isomorphic to $C_0(X)$ for some X.*

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Theorem (Marcoux–Popov, 2013 preprint)

Let A be a closed, commutative subalgebra of $\mathcal{B}(\mathsf{H})$. If A is (operator) amenable, then A is isomorphic to $C_0(X)$ for some X.

The strategy is to prove that the Gelfand transform $A \to C_0(\Phi_A)$ is bounded below. (From there the rest is a standard application of Sheinberg's theorem.)

Recall the theorem of Brannan and Samei: if G is a SIN group then every c.b. HM $\mathrm{A}(G) \to \mathcal{B}(\mathsf{H})$ is similar to a $*$-HM.

Now observe: if G is an amenable locally compact group, then $\mathrm{A}(G)$ is **operator amenable**.

So Marcoux and Popov's result has the following corollary.

Corollary

Let G be an **amenable** *locally compact group. Then every c.b. HM $\mathrm{A}(G) \to \mathcal{B}(\mathsf{H})$ is similar to a $*$-HM.*

Is there a LCQG proof of this, like the argument for the SIN case?

Question.

Let A be a **separable** closed subalgebra of $\mathcal{B}(\mathsf{H})$. If A is amenable, must it be isomorphic to a C^*-algebra?

Amenable subalgebras of $\mathcal{K}(\mathsf{H})$ are always isomorphic to C^*-algebras (Gifford, 1997/2006).

Question.

What if you replace $\mathcal{K}(\mathsf{H})$ with your favourite separable amenable C^*-algebra?

Question.

Let A be a **weak*-closed**, “Connes-amenable” subalgebra of $\mathcal{B}(\mathsf{H})$. Must it be isomorphic to a von Neumann algebra?