Higher-dimensional amenability of Banach algebras

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Amenability

B. E. Johnson (1972): $\mathcal{A}$ is **amenable**
$\iff$ for each Banach $\mathcal{A}$-bimodule X, every continuous derivation $D : \mathcal{A} \to X^*$ is inner, that is, $D(a) = a \cdot f - f \cdot a$ for some $f \in X^*$.

(Here $(a \cdot f)(x) = f(x \cdot a)$ and $(f \cdot a)(x) = f(a \cdot x)$ for $a \in \mathcal{A}$, $f \in X^*$, $x \in X$.)

$\iff$ the continuous Hochschild cohomology $\mathcal{H}^1(\mathcal{A}, X^*) = \{0\}$ for all Banach $\mathcal{A}$-bimodules X.

$\iff$ $\mathcal{H}^n(\mathcal{A}, X^*) = \{0\}$ for all $n \geq 1$ and for all Banach $\mathcal{A}$-bimodules X.

$\iff$ there exists a **virtual diagonal** $M \in (\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ such that for all $a \in \mathcal{A}$,

$$aM = Ma, \;\; \pi^{**}(M)a = a.$$

Here π is the product map on $\mathcal{A}$, $\hat{\otimes}$ is a notation for the projective tensor product of Banach spaces.

n-amenability of Banach algebras

The name **amenable** is used for such algebras because of the following theorem by B.E Johnson and J.R. Ringrose (1972):

A group algebra $L^1(G)$ is amenable $\iff$ the locally compact group G is amenable.

A.L.T Paterson [Pa96]: For $n \geq 1$, $\mathcal{A}$ is called n-**amenable** if $\mathcal{H}^n(\mathcal{A}, X^*) = \{0\}$ for every Banach $\mathcal{A}$-bimodule X.

Thus $\mathcal{A}$ is amenable $\iff$ $\mathcal{A}$ is 1-amenable.

It is clear that $\mathcal{A}$ is amenable $\Rightarrow$ $\mathcal{A}$ is n-amenable for all $n \geq 1$.

On amenability of operator algebras

A. Connes, U. Haagerup, G. Elliott: *Amenability* for C^*-algebras is equivalent to *nuclearity*.

For example, $C_0(\Omega)$ for any locally compact space Ω, and $\mathcal{K}(H)$ are amenable.

S. Wassermann: $\mathcal{B}(H)$ is not nuclear, so it is not amenable.

S.A. Argyros and R.G. Haydon (2009) constructed a Banach $\mathcal{L}_\infty$-space X such that $\mathcal{B}(X) = \mathcal{K}(X) \oplus \mathbf{C}$.

N. Gronbaek, B.E. Johnson and G.A. Willis (1994) proved that, for a Banach $\mathcal{L}_\infty$-space X, $\mathcal{K}(X)$ is amenable, hence $\mathcal{B}(X) = \mathcal{K}(X) \oplus \mathbf{C}$ is amenable too.

Open Problem. Describe infinite-dimensional Banach spaces E such that the Banach algebra $\mathcal{B}(E)$ is not amenable/ is amenable.

Note: it is known that $\mathcal{B}(l_p)$ is not amenable for $1 \leq p \leq \infty$ (S. Wassermann, C.J. Read, G. Pisier, N. Ozawa, V. Runde).

Examples of 2-amenable Banach algebras

1. Yu. Selivanov [He89; p.286] (see also A.L.T. Paterson [Pa96; Pages 179-180]) proved that the Banach algebra $\mathcal{A}$ of 2×2-complex matrices of the form

$$\begin{bmatrix} * & * \\ 0 & 0 \end{bmatrix}$$

is 2-amenable but not 1-amenable. The multiplication in $\mathcal{A}$ is determined by the products:

$$e^2 = e,\ ef = f \text{ and } fe = 0 = f^2.$$

Here

$$e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } f = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

We note that e is a left unit for $\mathcal{A}$, there is no identity. Thus A is not amenable.

Examples of 2-amenable Banach algebras

Claim: $\mathcal{A}$ is biprojective. A morphism of Banach $\mathcal{A}$-bimodules $\rho : \mathcal{A} \to \mathcal{A}\hat{\otimes}\mathcal{A}$ is given by

$$\rho(e) = e \otimes e \text{ and } \rho(f) = e \otimes f.$$

Therefore, for all Banach $\mathcal{A}$-bimodules X

$$\mathcal{H}^2(\mathcal{A}, X^*) = \{0\}.$$

Hence $\mathcal{A}$ is 2-amenable.

2. Let T_2 be the upper triangular 2×2-complex matrices. The algebra T_2 can be obtained by adjoining an identity to the algebra $\mathcal{A}$ from the previous example, and so it is 2-amenable, but not amenable.

Examples of n-amenable Banach algebras

In 1997 A.L.T. Paterson and R.R. Smith [PaSm97; Theorem 4.2] proved that the Banach algebra

$B_n = S^{n-1}(A_4)$ is $(n+1)$-amenable but not n-amenable.

Here A_4 is the Banach subalgebra of $M_4(\mathbb{C})$ of elements of the form

$$\begin{bmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ * & * & * & 0 \\ * & * & 0 & * \end{bmatrix}$$

The two-point suspension $S(A_4)$ of the algebra A_4 is the subalgebra of $B(\mathbb{C}^2 \oplus \mathbb{C}^4)$ whose elements are of the form

$$\begin{bmatrix} d & 0 \\ u & a \end{bmatrix}$$

$d \in \mathcal{D}_2$ the diagonal 2×2 complex matrices, $a \in A_4$, $u \in \mathcal{B}(\mathbb{C}^2, \mathbb{C}^4)$.

The Hochschild homology and cohomology groups of $\mathcal{A}$

Let $\mathcal{A}$ be a Banach algebra and X be a Banach $\mathcal{A}$-bimodule. The **continuous homology** $\mathcal{H}_n(\mathcal{A}, X)$ **of** $\mathcal{A}$ **with coefficients in** X is defined to be the nth homology

$$\mathcal{H}_n(\mathcal{C}_\sim(\mathcal{A}, X)) = \mathrm{Ker}\ b_{n-1}/\mathrm{Im}\ b_n$$

of the standard homological chain complex $(\mathcal{C}_\sim(\mathcal{A}, X))$:

$$0 \longleftarrow X \xleftarrow{b_0} X\hat{\otimes}\mathcal{A} \longleftarrow \ldots \longleftarrow X\hat{\otimes}\mathcal{A}^{\hat{\otimes}n} \xleftarrow{b_n} X\hat{\otimes}\mathcal{A}^{\hat{\otimes}(n+1)} \longleftarrow \ldots,$$

where the differentials b_* are given by

$$b_n(x \otimes a_1 \otimes \cdots \otimes a_{n+1}) = (x \cdot a_1 \otimes a_2 \otimes \cdots \otimes a_{n+1}) +$$

$$\sum_{i=1}^{n}(-1)^i(x \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}) + (-1)^{n+1}(a_{n+1} \cdot x \otimes a_1 \otimes \cdots \otimes a_n).$$

The Hochschild cohomology groups of $\mathcal{A}$ with coefficients in the dual $\mathcal{A}$-bimodule X^*

$$\mathcal{H}^n(\mathcal{A}, X^*) \cong H^n((\mathcal{C}_\sim(\mathcal{A}, X))^*)$$

the cohomology groups of the dual complex $(\mathcal{C}_\sim(\mathcal{A}, X))^*$.

$$\mathcal{H}^n(\mathcal{A}, X^*) \cong H^n((\mathcal{C}_\sim(\mathcal{A}, X))^*)$$

The dual of the standard homological chain complex $\mathcal{C}_\sim(\mathcal{A}, X)$

$$0 \longrightarrow X^* \xrightarrow{b_0^*} (X \hat{\otimes} \mathcal{A})^* \longrightarrow \ldots \longrightarrow (X \hat{\otimes} \mathcal{A}^{\hat{\otimes} n})^* \xrightarrow{b_n^*} (X \hat{\otimes} \mathcal{A}^{\hat{\otimes}(n+1)})^* \longrightarrow \ldots$$

The standard homological cochain complex $\mathcal{C}^\sim(\mathcal{A}, X^*)$ is

$$0 \longrightarrow X^* \xrightarrow{d^0} \mathcal{B}(\mathcal{A}, X^*) \longrightarrow \ldots \longrightarrow \mathcal{B}(\mathcal{A}^{\hat{\otimes} n}, X^*) \xrightarrow{d^n} \mathcal{B}(\mathcal{A}^{\hat{\otimes}(n+1)}, X^*) \longrightarrow \ldots$$

where

$$d^n f(a_1, \ldots, a_{n+1}) = a_1 \cdot f(a_2, \ldots, a_{n+1}) +$$

$$\sum_{k=1}^{n} (-1)^k f(a_1, \ldots, a_{k-1}, a_k a_{k+1}, \ldots, a_{n+1}) + (-1)^{n+1} f(a_1, \ldots, a_n) \cdot a_{n+1}.$$

One can show that the standard homological cochain complex $\mathcal{C}^{\sim}(\mathcal{A}, X^*)$ is isomorphic to the dual of $\mathcal{C}_{\sim}(\mathcal{A}, X)$. Note that, for all $n \geq 1$, there exists the following isometric isomorphism

$$\mathcal{B}(\mathcal{A}^{\hat{\otimes} n}, X^*) \cong (X \hat{\otimes} \mathcal{A}^{\hat{\otimes} n})^* : T \mapsto F$$

where

$$F(x \otimes a_1 \otimes \cdots \otimes a_n) = T(a_1 \otimes \cdots \otimes a_n)(x).$$

The rest follows.

n-amenability of $\mathcal{A}$ and derivations $D : \mathcal{A} \to Y^*$, where $Y^* = (X \hat{\otimes} \mathcal{A}^{\hat{\otimes}(n-1)})^*$

Proposition 1. *$\mathcal{A}$ is n-amenable $\iff$ for each Banach $\mathcal{A}$-bimodule X, every continuous derivation $D : \mathcal{A} \to Y^*$ where $Y^* = (X \hat{\otimes} \mathcal{A}^{\hat{\otimes}(n-1)})^*$ is inner, that is, $D(a) = a \cdot T - T \cdot a$ for some $T \in Y^*$.*

Here $\mathcal{A}$-module multiplication on Y^* depends on b^*_{n-1}:

$$(a \cdot T)(x \otimes a_1 \otimes \cdots \otimes a_{n-1}) = T(x \cdot a \otimes a_1 \otimes \cdots \otimes a_{n-1})$$

and

$$(T \cdot a)(x \otimes a_1 \otimes \cdots \otimes a_{n-1}) = T(x \otimes a \cdot a_1 \otimes \cdots \otimes a_{n-1})$$
$$+ \sum_{i=1}^{n-2} (-1)^i T(x \otimes a \otimes a_1 \otimes \cdots \otimes a_i \cdot a_{i+1} \otimes \cdots \otimes a_{n-1})$$
$$+ (-1)^{n-1} T(a_{n-1} \cdot x \otimes a \otimes a_1 \otimes \cdots \otimes a_{n-2})$$

for $a, a_1, \ldots, a_{n-1} \in \mathcal{A}$, $T \in Y^*$, $x \in X$.

n-amenability of $\mathcal{A}$

$\mathcal{A}$ is n-amenable $\iff$
$\iff$ $\mathcal{H}^p(\mathcal{A}, X^*) = \{0\}$ for all $p \geq n$ and for all Banach $\mathcal{A}$-bimodules X.

The last two statements were proved by Barry Johnson in 1972 [Jo72; Section 1.a]. It was shown there that, for positive integers $n \geq 1$ and $p \geq 1$,

$$\mathcal{H}^{n+p}(\mathcal{A}, X^*) \cong \mathcal{H}^n(\mathcal{A}, X_1^*)$$

where

$$X_1^* \cong \left(X \hat{\otimes} \mathcal{A}^{\hat{\otimes} p}\right)^*$$

$$\cong C^p(\mathcal{A}, X^*) = \{T : \mathcal{A}^{\hat{\otimes} p} \to X^*, \text{bounded linear operators}\}.$$

Here $\mathcal{A}$-module multiplication on X_1^* is defined by

$$(a \cdot F)(a_1 \otimes \cdots \otimes a_p) = a \cdot F(a_1 \otimes \cdots \otimes a_p)$$

and

$$(F \cdot a)(a_1 \otimes \cdots \otimes a_p) = F(a \cdot a_1 \otimes \cdots \otimes a_p)$$
$$+ \sum_{i=1}^{p-1} (-1)^i F(a \otimes a_1 \otimes \cdots \otimes a_i \cdot a_{i+1} \otimes \cdots \otimes a_p)$$
$$+ (-1)^p F(a \otimes a_1 \otimes \cdots \otimes a_{p-1}) \cdot a_p$$

for $a, a_1, \ldots, a_p \in \mathcal{A}$, $F \in X_1^*$.

The weak homological bidimension $db_w\mathcal{A}$ of $\mathcal{A}$

Yu.V. Selivanov [Se96]: The **weak homological bidimension** of a Banach algebra $\mathcal{A}$ is

$$db_w\mathcal{A} = \inf\ \{n : \mathcal{H}^{n+1}(\mathcal{A}, X^*) = \{0\} \text{ for all Banach } \mathcal{A}\text{−bimodule } X\}.$$

It is obvious that $\mathcal{A}$ is n-amenable $\iff db_w\mathcal{A} \leq n-1$.

$\mathcal{A}$ is n-amenable but not $(n-1)$-amenable $\iff db_w\mathcal{A} = n-1$.

$\mathcal{A}$ is amenable $\iff$ $\mathcal{A}$ is 1-amenable $\iff db_w\mathcal{A} = 0$.

Amenability and biflatness

A.Ya. Helemskii (1984) [He89]:

$\mathcal{A}$ is amenable $\iff$ $\mathcal{A}_+$ is biflat $\iff$ $\mathcal{A}_+$ is amenable

$\iff$ $\mathcal{A}$ is biflat and $\mathcal{A}$ has a bounded approximate identity.

Here $\mathcal{A}_+$ is the Banach algebra with the adjoined identity e.

A module $Y \in \mathcal{A}\text{-}\mathrm{mod}\text{-}\mathcal{A}$ is called *flat* if for any admissible complex $\mathcal{X}$ of Banach $\mathcal{A}$-bimodules

$$0 \longleftarrow X_0 \stackrel{\varphi_0}{\longleftarrow} X_1 \stackrel{\varphi_1}{\longleftarrow} X_2 \stackrel{\varphi_2}{\longleftarrow} X_3 \longleftarrow \cdots$$

the complex $\mathcal{X} \widehat{\otimes}_{\mathcal{A}-\mathcal{A}} Y$:

$$0 \longleftarrow X_0 \widehat{\otimes}_{\mathcal{A}-\mathcal{A}} Y \stackrel{\varphi_0 \widehat{\otimes}_{\mathcal{A}-\mathcal{A}} id_Y}{\longleftarrow} X_1 \widehat{\otimes}_{\mathcal{A}-\mathcal{A}} Y \stackrel{\varphi_1 \widehat{\otimes}_{\mathcal{A}-\mathcal{A}} id_Y}{\longleftarrow} X_2 \widehat{\otimes}_{\mathcal{A}-\mathcal{A}} Y \longleftarrow \cdots$$

is exact.

Here $\widehat{\otimes}_{\mathcal{A}-\mathcal{A}}$ is the projective tensor product of Banach $\mathcal{A}$-bimodules.

$db_w(\mathcal{A})$ of biflat Banach algebras

Theorem 1. [Se96, Theorem 6], [Se99] *Let $\mathcal{A}$ be a biflat Banach algebra. Then*
(i) *if $\mathcal{A}$ has two-sided b.a.i., then*

$$db_w(\mathcal{A}) = 0$$

and $\mathcal{A}$ is amenable;
(ii) *if $\mathcal{A}$ has left [right], but not two-sided b.a.i., then*

$$db_w(\mathcal{A}) = 1$$

and $\mathcal{A}$ is 2-amenable but not amenable;
(iii) *if $\mathcal{A}$ has neither left nor right b.a.i., then*

$$db_w(\mathcal{A}) = 2$$

and $\mathcal{A}$ is 3-amenable but not 2-amenable.

More examples of n-amenable Banach algebras

• The algebra $\mathcal{K}(l_2 \hat{\otimes} l_2)$ of compact operators on $l_2 \hat{\otimes} l_2$ is 2-amenable but not $(1$-$)$amenable, that is, $db_w \mathcal{K}(l_2 \hat{\otimes} l_2) = 1$.

In [GJW94] it was shown that the algebra $\mathcal{K}(l_2 \hat{\otimes} l_2)$ of compact operators on $l_2 \hat{\otimes} l_2$ does not have a right bounded approximate identity and therefore is not amenable. Selivanov proved in [Se02; Theorem 5.3.2] that $\mathcal{K}(l_2 \hat{\otimes} l_2)$ is biflat and $db_w \mathcal{K}(l_2 \hat{\otimes} l_2) = 1$.

• The algebra l_1 and the algebra $\mathcal{N}(H)$ of nuclear operators on an infinite-dimensional Hilbert space H are biprojective, and so biflat. They have neither left nor right b.a.i. Thus

$db_w l_1 = 2$ and $db_w \mathcal{N}(H) = 2$.

Therefore l_1 and $\mathcal{N}(H)$ are 3-amenable but not 2-amenable.

Infinite-dimensional Hilbert algebras are not n-amenable

Recall that $\mathcal{A}$ is a Hilbert algebra if it is a Banach *-algebra with respect to the norm $\|a\| = \langle a, a \rangle^{\frac{1}{2}}$ given by an inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{A}$ such that $aa^* \neq 0$ if $a \neq 0$, $\|a\| = \|a^*\|$ and $\langle ab, c \rangle = \langle b, a^*c \rangle$ for all $a, b, c \in \mathcal{A}$.

Theorem 2. [Se02] *Let $\mathcal{A}$ be an infinite-dimensional Hilbert algebra. Then, for each $n = 1, 2, \ldots,$*

$$\mathcal{H}^{2n}(\mathcal{A}, \mathbb{C}) \neq \{0\}$$

where $\mathbb{C}$ is the trivial $\mathcal{A}$-bimodule and therefore

$$db_w \mathcal{A} = \infty.$$

Hint: Show that

$$f(a_1, \ldots, a_{2n}) = \langle a_1, a_2^* \rangle \langle a_3, a_4^* \rangle \ldots \langle a_{2n-1}, a_{2n}^* \rangle, \; a_1, \ldots, a_{2n} \in \mathcal{A},$$

is $2n$-cocycle, but $f \neq d^{2n-1}(g)$ for any $g \in C^{2n-1}(\mathcal{A}, \mathbb{C})$.

Examples of Hilbert algebras:

l_2 with coordinatewise multiplication,

the algebra $\mathcal{HS}(H)$ of Hilbert-Schmidt operators on a Hilbert space H.

All infinite-dimensional Hilbert algebras are not n-amenable for any $n \geq 1$, and so $db_w l_2 = \infty$ and $db_w \mathcal{HS}(H) = \infty$ for an infinite-dimensional H.

Questions

Question. Are there C^*-algebras which are 2-amenable but not amenable? which are 3-amenable but not 2-amenable?

Question. Are there Fourier algebras $A(G)$ which are 2-amenable but not amenable? which are 3-amenable but not 2-amenable?

Is there a locally compact group G such that $A(G)$ is biflat, but not amenable?

The following is known:

H. Leptin: $A(G)$ has b.a.i $\iff$ G is amenable.

B.E. Forrest and V. Runde (2005): $A(G)$ is amenable $\iff$ G admits an abelian subgroup of finite index.

V. Runde (2009): If $A(G)$ is biflat then either (a) G admits an abelian subgroup of finite index, or (b) G is non-amenable and does not contain a discrete copy of the free group of two generators.

Relations between triviality of homology and cohomology groups

Proposition 2. *Let $(\mathcal{X}, d)$ be a chain complex of Fréchet spaces and continuous linear operators and let $N \in \mathbf{N}$. Then the following statements are equivalent:*

(i) $H_n(\mathcal{X}, d) = \{0\}$ *for all* $n \geq N$ *and* $H_{N-1}(\mathcal{X}, d)$ *is Hausdorff;*

(ii) $H^n(\mathcal{X}^*, d^*) = \{0\}$ *for all* $n \geq N$*.*

Proof Recall that $H_n(\mathcal{X}, d) = \mathrm{Ker}\ d_{n-1}/\mathrm{Im}\ d_n$ and $H^n(\mathcal{X}^*, d^*) = \mathrm{Ker}\ d_n^*/\mathrm{Im}\ d_{n-1}^*$. Let L be the closure of $\mathrm{Im}\ d_{N-1}$ in X_{N-1}. Consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \leftarrow & L & \stackrel{j}{\longleftarrow} & X_N & \stackrel{d_N}{\longleftarrow} & X_{N+1} & \stackrel{d_{N+1}}{\longleftarrow} & \ldots \\ & & \downarrow i & \swarrow d_{N-1} & & & & & \\ & & X_{N-1} & & & & & & \end{array} \qquad (1)$$

in which i is the natural inclusion and j is a corestriction of d_{N-1}. The dual commutative diagram is the following

$$\begin{array}{ccccccccc} 0 & \to & L^* & \xrightarrow{j^*} & X_N^* & \xrightarrow{d_N^*} & X_{N+1}^* & \xrightarrow{d_{N+1}^*} & \dots \\ & & \uparrow i^* & \nearrow d_{N-1}^* & & & & & \\ & & X_{N-1}^* & & & & & & \end{array} \tag{2}$$

It is clear that $H_{N-1}(\mathcal{X}, d)$ is Hausdorff if and only if j is surjective. Since i is injective, condition (i) is equivalent to the exactness of the 1st line of diagram (1). On the other hand, by the Hahn-Banach theorem, i^* is surjective. It is also clear that j^* is injective. Thus condition (ii) is equivalent to the exactness of the 1st line of diagram (2).

In the case of Fréchet spaces, the exactness of the 1st line of the complex (1) is equivalent to the exactness of the 1st line of the complex (2), see [Ly06; Lemma 2.3].

The proposition is proved.

Fréchet algebras of finite homological bidimension

Recall that the homological dimension of A_+ in the category of Fréchet $\mathcal{A}$-bimodules is called the *homological bidimension* of $\mathcal{A}$. It is denoted by $\mathrm{db}\ \mathcal{A}$; see [He89; Def. 3.5.9].

If $\mathrm{db}\ \mathcal{A} = m$, by [He89; Theorem 3.4.25], for any Fréchet $\mathcal{A}$-bimodule X, the homology groups $\mathcal{H}_n(\mathcal{A}, X) = \mathrm{Tor}_n^{\mathcal{A}^e}(X, \mathcal{A}_+) = \{0\}$ for all $n \geq m+1$ and $\mathcal{H}_m(\mathcal{A}, X) = \mathrm{Tor}_m^{\mathcal{A}^e}(X, \mathcal{A}_+)$ is Hausdorff. Hence $\mathrm{db}_w\ \mathcal{A} \leq \mathrm{db}\ \mathcal{A} = m$.

Here $\mathcal{A}^e = \mathcal{A}_+ \hat{\otimes} \mathcal{A}_+^{op}$ is the enveloping algebra of $\mathcal{A}$, $\mathcal{A}_+^{op}$ is the opposite algebra of $\mathcal{A}_+$ with multiplication $a \cdot b = ba$.

Examples of Fréchet algebras of finite homological bidimension

(i) Let $\mathcal{O}(\mathcal{U})$ be the Fréchet algebra of holomorphic functions on a polydomain $U = U_1 \times U_2 \times \ldots U_m \subseteq \mathbf{C}^m$. Then $\mathrm{db}\ \mathcal{O}(\mathcal{U})\ = m$ [Ta72].

(ii) Let M be any infinitely smooth manifold of topological dimension m, and let $C^\infty(M)$ be the Fréchet algebra of all infinitely smooth functions on M. Then $\mathrm{db}\, C^\infty(M) = m$ [Og86]. For relations between the continuous cyclic cohomology of $C^\infty(M)$ and de Rham homology of M, see [Co94].

(iii) Let $\mathcal{S}(\mathbf{R}^m)$ be the Fréchet algebra of rapidly decreasing infinitely smooth functions on $\mathbf{R}^m$. Then $\mathrm{db}\ \mathcal{S}(\mathbf{R}^m) = m$ [OgHe84].

Excision in the cohomology of Banach algebras with coefficients in dual modules

Theorem 3. [LW98] *Let*

$$0 \to I \xrightarrow{i} A \xrightarrow{j} A/I \to 0$$

be an extension of Banach algebras and let X be a Banach A-bimodule such that $\overline{IX} = \overline{XI}$. Suppose I has a bounded approximate identity. Then there exists an associated long exact sequence of continuous cohomology groups

$$0 \to \mathcal{H}^0(A/I, (X/\overline{IX})^*) \to \ldots \mathcal{H}^{n-1}(I, (\overline{IX})^*) \to$$
$$\mathcal{H}^n(A/I, (X/\overline{IX})^*) \to \mathcal{H}^n(A, X^*) \to \mathcal{H}^n(I, (\overline{IX})^*) \to \ldots .$$

Ideals with b.a.i and essential modules

Proposition 3. [LW98] *Let $\mathcal{A}$ be a Banach algebra and let I be a closed two-sided ideal of $\mathcal{A}$. Suppose that I has a b.a.i. Then,*

(i) *for any Banach I-bimodule Z,*

$$\mathcal{H}_n(I, Z) = \mathcal{H}_n(\mathcal{A}, \overline{IZI}) \text{ and } \mathcal{H}^n(I, Z^*) = \mathcal{H}^n(\mathcal{A}, (\overline{IZI})^*) \text{ for all } n \geq 1.$$

(ii) *for any Banach A/I-bimodule Y,*

$$\mathcal{H}_n(A/I, Y) = \mathcal{H}_n(A, Y) \text{ and } \mathcal{H}^n(A/I, Y^*) = \mathcal{H}^n(A, Y^*) \text{ for all } n \geq 0.$$

Remark 4. *Proposition 3 shows that in the case of Banach algebras $\mathcal{A}$ with b.a.i. we can restrict ourselves to the category of* essential Banach modules *in questions on db_w and $\mathcal{H}^n(\mathcal{A}, X^*)$.*

$db_w\mathcal{A} \geq \max\{db_w I, db_w \mathcal{A}/I\}$ for a closed two-sided ideal I with a b.a.i.

Theorem 5. [LW98] *Let $\mathcal{A}$ be a Banach algebra and let I be a closed two-sided ideal of $\mathcal{A}$. Suppose that I has a b.a.i.. Then*

(i) *the n-amenability of A implies the n-amenability of the two Banach algebras A/I and I;*

(ii) $db_w I \leq db_w A$ *and* $db_w A/I \leq db_w A$.

$db_w \mathcal{A} \widehat{\otimes} \mathcal{B}$

B.E. Johnson (1972):
If Banach algebras $\mathcal{A}$ and $\mathcal{B}$ are amenable then their tensor product $\mathcal{A} \widehat{\otimes} \mathcal{B}$ is amenable too.

Suppose $db_w \mathcal{A} = m$ and $db_w \mathcal{B} = q$. Question: *What can we say about the higher-dimensional amenability of* $\mathcal{A} \widehat{\otimes} \mathcal{B}$*?*

In 1996 Yu. Selivanov remarked without proof that, for $\mathcal{A}$ and $\mathcal{B}$ with bounded approximate identities,

$$db_w \mathcal{A} \widehat{\otimes} \mathcal{B} = db_w \mathcal{A} + db_w \mathcal{B}.$$

In 2002 he gave a proof of the formula in the particular case of algebras with identities and his proof depends heavily on the existence of identities.

We prove that the formula

$$db_w \mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{B} = db_w \mathcal{A} + db_w \mathcal{B}$$

is correct for algebras $\mathcal{A}$ and $\mathcal{B}$ b.a.i..

– We show further that the formula does **not** hold for algebras with no b.a.i, nor for algebras with only 1-sided b.a.i.

– The well-known trick adjoining of an identity to the algebra does not work for the tensor product of algebras.

The homological properties of the tensor product algebras $\mathcal{A}\widehat{\otimes}\mathcal{B}$ and $\mathcal{A}_+\widehat{\otimes}\mathcal{B}_+$ are different.

n-amenability and flat resolutions of $\mathcal{A}_+$

Theorem 6. [Se96] *Let $\mathcal{A}$ be a Banach algebra. For each integer $n \geq 0$ the following properties of $\mathcal{A}$ are equivalent:*

(i) $db_w\mathcal{A} \leq n$;

(ii) $\mathcal{A}$ *is* $(n+1)$*-amenable, that is,* $\mathcal{H}^{n+1}(\mathcal{A}, X^*) = \{0\}$ for all $X \in \mathcal{A}\text{-mod-}\mathcal{A}$;

(iii) *the* $\mathcal{A}$*-bimodule* $\mathcal{A}_+$ *has a flat admissible resolution of length* n *in the category of* $\mathcal{A}\text{-mod-}\mathcal{A}$;

(iv) *if* $0 \longleftarrow \mathcal{A}_+ \overset{\varepsilon}{\longleftarrow} P_0 \overset{\varphi_0}{\longleftarrow} P_1 \overset{\varphi_1}{\longleftarrow} \cdots P_{n-1} \overset{\varphi_{n-1}}{\longleftarrow} Y \longleftarrow 0 \qquad (0 \leftarrow \mathcal{A} \leftarrow \mathcal{P})$

is an admissible resolution of $\mathcal{A}_+$ *in which all the modules* P_i *are flat in* $\mathcal{A}\text{-mod-}\mathcal{A}$*, then* Y *is also flat in* $\mathcal{A}\text{-mod-}\mathcal{A}$*.*

It is well known that $db_w\mathcal{A} = db_w\mathcal{A}_+$.

Pseudo-resolutions in categories of Banach modules $\mathcal{K}$

Definition 1. *For $X \in \mathcal{K}$, a complex*

$$0 \longleftarrow X \overset{\varepsilon}{\longleftarrow} Q_0 \overset{\varphi_0}{\longleftarrow} Q_1 \overset{\varphi_1}{\longleftarrow} Q_2 \longleftarrow \ldots$$

is called a $pseudo\text{-}resolution\ of\ X\ in\ \mathcal{K}$ if it is weakly admissible,

and a $flat\ pseudo\text{-}resolution\ of\ X\ in\ \mathcal{K}$ if, in addition, all the modules in $\mathcal{Q}$ are flat in $\mathcal{K}$.

A complex of Banach modules is called weakly admissible if its dual complex splits as a complex of Banach spaces.

Flat pseudo-resolution of $\mathcal{A}$ with b.a.i. in $\mathcal{A}$-mod-$\mathcal{A}$

We put $\beta_n(\mathcal{A}) = \mathcal{A}^{\widehat{\otimes}^{n+2}}$, $n \geq 0$, and let $d_n : \beta_{n+1}(\mathcal{A}) \to \beta_n(\mathcal{A})$ be given by

$$d_n(a_0 \otimes \ldots \otimes a_{n+2}) =$$

$$\sum_{k=0}^{n+1} (-1)^k (a_0 \otimes \ldots \otimes a_k a_{k+1} \otimes \ldots \otimes a_{n+2}).$$

One can prove that the complex

$$0 \leftarrow \mathcal{A} \xleftarrow{\pi} \beta_0(\mathcal{A}) \xleftarrow{d_0} \beta_1(\mathcal{A}) \xleftarrow{d_1} \cdots \leftarrow \beta_n(\mathcal{A}) \xleftarrow{d_n} \beta_{n+1}(\mathcal{A}) \leftarrow \ldots,$$

where $\pi : \beta_0(\mathcal{A}) \to \mathcal{A} : a \otimes b \mapsto ab$, is a flat pseudo-resolution of the $\mathcal{A}$-bimodule $\mathcal{A}$. We denote it by $0 \leftarrow \mathcal{A} \xleftarrow{\pi} \beta(\mathcal{A})$.

Examples weakly admissible extensions which are not admissible

For instance, in the extension of C^*-algebras

$$0 \to c_0 \to \ell^\infty \to \ell^\infty / c_0 \to 0$$

the closed ideal c_0 of sequences convergent to zero is not complemented in the C^*-algebra ℓ^∞ of bounded sequences with componentwise multiplication (Phillips' Lemma).

For an infinite-dimensional Hilbert space H, the extension

$$0 \to \mathcal{K}(H) \to \mathcal{B}(H) \to \mathcal{B}(H)/\mathcal{K}(H) \to 0,$$

where $\mathcal{K}(H)$ is the closed ideal of compact operators in the C^*-algebra $\mathcal{B}(H)$ of bounded operators, is not admissible (J. B. Conway).

Thus these extensions are not admissible, but they are weakly admissible.

Topologically pure extensions

Definition 2. *A short exact sequence of Fréchet spaces and continuous operators*

$$0 \to Y \xrightarrow{i} Z \xrightarrow{j} W \to 0$$

is called topologically pure in $\mathcal{F}r$ if for every $X \in \mathcal{F}r$ the sequence

$$0 \to X \hat{\otimes} Y \xrightarrow{\mathrm{id}_X \hat{\otimes} i} X \hat{\otimes} Z \xrightarrow{\mathrm{id}_X \hat{\otimes} j} X \hat{\otimes} W \to 0$$

is exact.

An extension of Fréchet algebras is called topologically pure if the underlying short exact sequence of Fréchet spaces is topologically pure in $\mathcal{F}r$.

Admissible extensions and topologically pure extensions

Lemma 1. *(J. Cigler, V. Losert and P. W. Michor [CLM]) A short exact sequence of Banach spaces and continuous linear operators*

$$0 \to Y \xrightarrow{i} Z \xrightarrow{j} W \to 0 \tag{3}$$

is weakly admissible in $\mathcal{B}an$ *if and only if it is topologically pure in* $\mathcal{B}an$.

The reason for the introduction of topologically pure extensions

They allow

– circumvention of the known problem that the projective tensor product of injective continuous linear operators is not necessarily injective,

– as well as ensuring that $(\mathrm{id}_X \otimes i)(X \hat{\otimes} Y)$ is closed in $X \hat{\otimes} Z$.

It can be proved that extensions of Fréchet spaces which satisfy one of the following conditions are topologically pure:

(i) an admissible extension, that is, one which admits a continuous linear splitting;

(ii) a weakly admissible extension, that is, one whose strong dual sequence is admissible;

(iii) an extension of nuclear Fréchet spaces (see, for example, J. Eschmeier and M. Putinar, and J. L. Taylor);

(iv) an extension of Fréchet algebras such that Y has a left or right bounded approximate identity.

$db_w\mathcal{A} \leq n$ and pseudo-resolutions for Banach algebras $\mathcal{A}$ with a b.a.i.

Theorem 7. [Jo72] plus [Ly12] *Let $\mathcal{A}$ be a Banach algebra with b.a.i.. For each integer $n \geq 0$ the following properties of $\mathcal{A}$ are equivalent:*

(i) $db_w\mathcal{A} \leq n$;

(ii) $\mathcal{A}$ *is* $(n+1)$*-amenable, that is,* $\mathcal{H}^{n+1}(\mathcal{A}, X^*) = \{0\}$ for all $X \in \mathcal{A}$-mod-$\mathcal{A}$;

(iii) $\mathcal{H}^m(\mathcal{A}, X^*) = \{0\}$ for all $m \geq n+1$ and for all $X \in \mathcal{A}$-essmod-$\mathcal{A}$;

(iv) $\mathcal{H}_{n+1}(\mathcal{A}, X) = \{0\}$ *and* $\mathcal{H}_n(\mathcal{A}, X)$ *is a Hausdorff space for all* $X \in \mathcal{A}$-essmod-$\mathcal{A}$;

(v) *if* $0 \longleftarrow \mathcal{A} \xleftarrow{\varepsilon} P_0 \xleftarrow{\varphi_0} P_1 \xleftarrow{\varphi_1} \cdots P_{n-1} \xleftarrow{\varphi_{n-1}} Y \longleftarrow 0 \qquad (0 \leftarrow \mathcal{A} \leftarrow \mathcal{P})$

is a pseudo-resolution of $\mathcal{A}$ *in* $\mathcal{A}$-essmod-$\mathcal{A}$ *such that all the modules* P_i *are flat in* $\mathcal{A}$-essmod-$\mathcal{A}$, *then* Y *is also flat in* $\mathcal{A}$-essmod-$\mathcal{A}$.

(vi) *the* $\mathcal{A}$*-bimodule* $\mathcal{A}$ *has a flat pseudo-resolution of length* n *in the category of* $\mathcal{A}$-essmod-$\mathcal{A}$.

$$db_w(\mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{B}) \geq \max\{db_w\mathcal{A}, db_w\mathcal{B}\}$$

Proposition 4. ([Se02] plus [Ly12]) *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras with b.a.i.. Then*

$$db_w(\mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{B}) \geq \max\{db_w\mathcal{A}, db_w\mathcal{B}\}.$$

Questions. Is it true that for all Banach algebras $\mathcal{A}$ and $\mathcal{B}$

$$db_w(\mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{B}) \leq db_w\mathcal{A} + db_w\mathcal{B}?$$

$$db_w(\mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{B}) \geq \max\{db_w\mathcal{A}, db_w\mathcal{B}\}?$$

The tensor product $\mathcal{X} \widehat{\otimes} \mathcal{Y}$ of bounded complexes

Definition 3. *Let $\mathcal{X}$, $\mathcal{Y}$ be chain complexes in $\mathcal{B}an$:*

$$0 \xleftarrow{\varphi_{-1}} X_0 \xleftarrow{\varphi_0} X_1 \xleftarrow{\varphi_1} X_2 \xleftarrow{\varphi_2} X_3 \longleftarrow \cdots$$

and

$$0 \xleftarrow{\psi_{-1}} Y_0 \xleftarrow{\psi_0} Y_1 \xleftarrow{\psi_1} Y_2 \xleftarrow{\psi_2} Y_3 \longleftarrow \cdots .$$

The tensor product $\mathcal{X} \widehat{\otimes} \mathcal{Y}$ of bounded complexes $\mathcal{X}$ and $\mathcal{Y}$ in $\mathcal{B}an$ is the chain complex

$$0 \xleftarrow{\delta_{-1}} (\mathcal{X} \widehat{\otimes} \mathcal{Y})_0 \xleftarrow{\delta_0} (\mathcal{X} \widehat{\otimes} \mathcal{Y})_1 \xleftarrow{\delta_1} (\mathcal{X} \widehat{\otimes} \mathcal{Y})_2 \longleftarrow \cdots , \tag{4}$$

where

$$(\mathcal{X}\widehat{\otimes}\mathcal{Y})_n = \bigoplus_{m+q=n} X_m \widehat{\otimes} Y_q$$

and

$$\delta_{n-1}(x \otimes y) = \varphi_{m-1}(x) \otimes y + (-1)^m x \otimes \psi_{q-1}(y),$$

$x \in X_m, y \in Y_q$ *and* $m + q = n$.

The tensor product algebra $\mathcal{A} \widehat{\otimes} \mathcal{B}$ of biflat Banach algebras $\mathcal{A}$ and $\mathcal{B}$ is biflat.

Proposition 5. [Ly12] *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras, let X be an essential Banach $\mathcal{A}$-bimodule and let Y be an essential Banach $\mathcal{B}$-bimodule. Suppose X is flat in $\mathcal{A}$-mod-$\mathcal{A}$ and Y is flat in $\mathcal{B}\text{-mod-}\mathcal{B}$. Then $X \widehat{\otimes} Y$ is flat in $\mathcal{A} \widehat{\otimes} \mathcal{B}\text{-mod-}\mathcal{A} \widehat{\otimes} \mathcal{B}$.*

As a corollary we have the following result.

Theorem 8. *The tensor product algebra $\mathcal{A} \widehat{\otimes} \mathcal{B}$ of biflat Banach algebras $\mathcal{A}$ and $\mathcal{B}$ is biflat.*

Proof. A biflat Banach algebra is essential (Helemskii). Hence, by Proposition 5, $\mathcal{A} \widehat{\otimes} \mathcal{B}$ is flat in $\mathcal{A} \widehat{\otimes} \mathcal{B}\text{-mod-}\mathcal{A} \widehat{\otimes} \mathcal{B}$.

The tensor product of flat pseudo-resolutions

Proposition 6. [Ly12] *Let* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *be Banach algebras.*

Let $0 \leftarrow X \stackrel{\varepsilon_1}{\longleftarrow} \mathcal{X}$ *be a pseudo-resolution of* X *in* $\mathcal{A}_1\text{-essmod-}\mathcal{A}_1$ *such that all modules in* $\mathcal{X}$ *are flat in* $\mathcal{A}_1\text{-mod-}\mathcal{A}_1$ *and*

$0 \leftarrow Y \stackrel{\varepsilon_2}{\longleftarrow} \mathcal{Y}$ *be a pseudo-resolution of* Y *in* $\mathcal{A}_2\text{-essmod-}\mathcal{A}_2$ *such that all modules in* $\mathcal{Y}$ *are flat in* $\mathcal{A}_2\text{-mod-}\mathcal{A}_2$*.*

Then $0 \leftarrow X \widehat{\otimes} Y \stackrel{\varepsilon_1 \otimes \varepsilon_2}{\longleftarrow} \mathcal{X} \widehat{\otimes} \mathcal{Y}$ *is a pseudo-resolution of* $X \widehat{\otimes} Y$ *in* $\mathcal{A}_1 \widehat{\otimes} \mathcal{A}_2\text{-essmod-}\mathcal{A}_1 \widehat{\otimes} \mathcal{A}_2$ *such that all modules in* $\mathcal{X} \widehat{\otimes} \mathcal{Y}$ *are flat in* $\mathcal{A}_1 \widehat{\otimes} \mathcal{A}_2\text{-mod-}\mathcal{A}_1 \widehat{\otimes} \mathcal{A}_2$*.*

$db_w\mathcal{A} < n$ for $\mathcal{A}$ with b.a.i.

Recall that a continuous linear operator $T : X \to Y$ between Banach spaces X and Y is *topologically injective* if it is injective and its image is closed, that is, $T : X \to \mathrm{Im}\ T$ is a topological isomorphism.

Proposition 7. ([Se02] plus [Ly12]) *Let $\mathcal{A}$ be a Banach algebra with b.a.i. and let $(0 \leftarrow \mathcal{A} \leftarrow \mathcal{P})$:*

$$0 \longleftarrow \mathcal{A} \stackrel{\varepsilon}{\longleftarrow} P_0 \stackrel{\varphi_0}{\longleftarrow} \cdots P_{n-1} \stackrel{\varphi_{n-1}}{\longleftarrow} P_n \longleftarrow 0 \tag{5}$$

be a flat pseudo-resolution of $\mathcal{A}$ in $\mathcal{A}$-essmod-$\mathcal{A}$. Then

$$db_w\mathcal{A} < n \quad \Longleftrightarrow$$

for every X in $\mathcal{A}$-essmod-$\mathcal{A}$, the operator

$$\varphi_{n-1} \otimes_{\mathcal{A}-\mathcal{A}} \mathrm{id}\ _{\mathrm{X}} : \mathrm{P_n} \widehat{\otimes}_{\mathcal{A}-\mathcal{A}} \mathrm{X} \to \mathrm{P_{n-1}} \widehat{\otimes}_{\mathcal{A}-\mathcal{A}} \mathrm{X}$$

is topologically injective.

Lemma 2. [Se96] *Let E_0, E, F_0 and F be Banach spaces, and let $S : E_0 \to E$ and $T : F_0 \to F$ be continuous linear operators. Suppose S and T are not topologically injective. Then the continuous linear operator*

$$\Delta : E_0 \widehat{\otimes} F_0 \to (E_0 \widehat{\otimes} F) \oplus (E \widehat{\otimes} F_0)$$

defined by

$$\Delta(x \otimes y) = (x \otimes T(y), S(x) \otimes y) \ \ (x \in E_0, y \in F_0).$$

is not topologically injective.

$db_w(\mathcal{A}\widehat{\otimes}\mathcal{B})$ for Banach algebras $\mathcal{A}$ and $\mathcal{B}$ with b.a.i.

Theorem 9. [Ly12] *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras with bounded approximate identities. Then*

$$db_w(\mathcal{A}\widehat{\otimes}\mathcal{B}) = db_w\mathcal{A} + db_w\mathcal{B}.$$

Proof. Suppose $db_w\mathcal{A} = m$ and $db_w\mathcal{B} = q$ where $0 < m, q < \infty$.

By Theorem 7, $db_w\mathcal{A} = m$ implies there is a flat pseudo-resolution

$$0 \leftarrow \mathcal{A} \xleftarrow{\varepsilon_1} (\mathcal{P}, \varphi)$$

of length m in the category $\mathcal{A}\text{-essmod-}\mathcal{A}$.

By Proposition 7, there exists $X \in \mathcal{A}\text{-essmod-}\mathcal{A}$ such that the operator

$$\varphi_{m-1} \otimes_{\mathcal{A}-\mathcal{A}} \mathrm{id}_{\mathrm{X}} : \mathrm{P_m}\widehat{\otimes}_{\mathcal{A}-\mathcal{A}}\mathrm{X} \to \mathrm{P_{m-1}}\widehat{\otimes}_{\mathcal{A}-\mathcal{A}}\mathrm{X}$$

is not topologically injective.

Similarly, $db_w\mathcal{B} = q$ implies that there is a flat pseudo-resolution

$$0 \leftarrow \mathcal{B} \xleftarrow{\varepsilon_2} (\mathcal{Q}, \psi)$$

of length q in the category $\mathcal{B}$-essmod-$\mathcal{B}$ and there exist $Y \in \mathcal{B}$-essmod-$\mathcal{B}$ such that the operator

$$\psi_{q-1} \otimes_{\mathcal{B}-\mathcal{B}} \mathrm{id}_{\mathrm{Y}} : \mathrm{Q_q} \widehat{\otimes}_{\mathcal{B}-\mathcal{B}} \mathrm{Y} \to \mathrm{Q_{q-1}} \widehat{\otimes}_{\mathcal{B}-\mathcal{B}} \mathrm{Y}$$

is not topologically injective.

By Proposition 6,

$$0 \leftarrow \mathcal{A} \widehat{\otimes} \mathcal{B} \xleftarrow{\varepsilon_1 \otimes \varepsilon_2} (\mathcal{P} \widehat{\otimes} \mathcal{Q}, \delta)$$

is a flat pseudo-resolution of $\mathcal{A} \widehat{\otimes} \mathcal{B}$ in $\mathcal{A} \widehat{\otimes} \mathcal{B}$-essmod-$\mathcal{A} \widehat{\otimes} \mathcal{B}$ of length $m + q$.

Take $Z = X \widehat{\otimes} Y$ in $\mathcal{A}\widehat{\otimes}\mathcal{B}\text{-essmod-}\mathcal{A}\widehat{\otimes}\mathcal{B}$.

By Lemma 2, the operator

$$\delta_{m+q-1} \otimes_{\mathcal{A}\widehat{\otimes}\mathcal{B}-\mathcal{A}\widehat{\otimes}\mathcal{B}} \mathrm{id}_{\,Z} : (\mathcal{P}\widehat{\otimes}\mathcal{Q})_{\mathrm{m+q}} \widehat{\bigotimes}_{\mathcal{A}\widehat{\otimes}\mathcal{B}-\mathcal{A}\widehat{\otimes}\mathcal{B}} \mathrm{Z}$$

$$\to (\mathcal{P}\widehat{\otimes}\mathcal{Q})_{m+q-1} \widehat{\bigotimes}_{\mathcal{A}\widehat{\otimes}\mathcal{B}-\mathcal{A}\widehat{\otimes}\mathcal{B}} Z$$

is not topologically injective. Therefore, by Proposition 7,

$$db_w(\mathcal{A}\widehat{\otimes}\mathcal{B}) = m + q.$$

Corollary 1. *Let $\mathcal{A}$ be an amenable Banach algebra and $\mathcal{B}$ be Banach algebras with b.a.i. Then*

(i)

$$db_w(\mathcal{A}\widehat{\otimes}\mathcal{B}) = db_w\mathcal{B}.$$

(ii) $\mathcal{A}\widehat{\otimes}\mathcal{B}$ *is* n*-amenable* $\iff$ $\mathcal{B}$ *is* n*-amenable.*

$db_w(\mathcal{A}\widehat{\otimes}\mathcal{B})$ for biflat $\mathcal{A}$ and $\mathcal{B}$

Theorem 10. [Ly12] *Let $\mathcal{A}$ and $\mathcal{B}$ be biflat Banach algebras. Then*
(i)

$$db_w(\mathcal{A}\widehat{\otimes}\mathcal{B}) = 0 \ \text{ and}$$
$$db_w(\mathcal{A}_+\widehat{\otimes}\mathcal{B}_+) = db_w\mathcal{A} + db_w\mathcal{B} = 0$$

if $\mathcal{A}$ and $\mathcal{B}$ have two-sided b.a.i.;
(ii)

$$db_w(\mathcal{A}\widehat{\otimes}\mathcal{B}) \leq 1 \ \text{ and}$$
$$db_w(\mathcal{A}_+\widehat{\otimes}\mathcal{B}_+) = db_w\mathcal{A} + db_w\mathcal{B} = 2$$

if $\mathcal{A}$ and $\mathcal{B}$ have left [right], but not two-sided b.a.i.;
(iii)

$$db_w(\mathcal{A}\widehat{\otimes}\mathcal{B}) \leq 2 \ \text{ and}$$
$$db_w(\mathcal{A}_+\widehat{\otimes}\mathcal{B}_+) = db_w\mathcal{A} + db_w\mathcal{B} = 4$$

if $\mathcal{A}$ and $\mathcal{B}$ have neither left nor right b.a.i.

The algebra $\mathcal{K}(\ell_2 \widehat{\otimes} \ell_2)$ of compact operators

Example 1. *The algebra $\mathcal{K}(\ell_2 \widehat{\otimes} \ell_2)$ of compact operators on $\ell_2 \widehat{\otimes} \ell_2$ is a biflat Banach algebra with a left, but not two-sided bounded approximate identity. By Theorem 10, for $n \geq 1$,*

$$db_w [\mathcal{K}(\ell_2 \widehat{\otimes} \ell_2)]^{\widehat{\otimes} n} \leq 1$$

and

$$db_w [\mathcal{K}(\ell_2 \widehat{\otimes} \ell_2)_+]^{\widehat{\otimes} n} = n.$$

The tensor algebra $E\widehat{\otimes}F$ generated by the duality $(E, F, \langle\cdot,\cdot\rangle)$

Example 2. *Let (E, F) be a pair of infinite-dimensional Banach spaces endowed with a nondegenerate jointly continuous bilinear form $\langle\cdot,\cdot\rangle : E \times F \to \mathbf{C}$ that is not identically zero. The space $\mathcal{A} = E\widehat{\otimes}F$ is a Banach algebra with respect to the multiplication defined by*

$$(x_1 \otimes x_2)(y_1 \otimes y_2) = \langle x_2, y_1\rangle x_1 \otimes y_2,\ x_i \in E,\ y_i \in F.$$

Then $\mathcal{A} = E\widehat{\otimes}F$ is called the $tensor\ algebra\ generated\ by\ the\ duality$ $(E, F, \langle\cdot,\cdot\rangle)$.

It is known that $\mathcal{A}$ is biprojective (Yu.V. Selivanov), and has neither a left nor a right b.a.i. (A. Grothendieck).

In particular, if E is a Banach space with the approximation property, then the algebra $\mathcal{A} = E\widehat{\otimes}E^*$ is isomorphic to the algebra $\mathcal{N}(E)$ of nuclear operators on E.

By Theorem 10, for $n \geq 1$,

$$db_w[E\widehat{\otimes}F]^{\widehat{\otimes}n} \leq 2 \ \text{ and } \ db_w[(E\widehat{\otimes}F)_+]^{\widehat{\otimes}n} = 2n.$$

Example 3. *Let $\mathcal{B}$ be the algebra of 2×2-complex matrices of the form*

$$\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$$

with matrix multiplication and norm. It is known that $\mathcal{B}$ is 2-amenable, biprojective, has a left, but not right identity. By Theorem 10, for $n \geq 1$,

$$db_w[\mathcal{B}]^{\widehat{\otimes} n} = 1, \text{ and } db_w[\mathcal{B}_+]^{\widehat{\otimes} n} = n;$$

$$db_w[\mathcal{B}\widehat{\otimes}\mathcal{K}(\ell_2\widehat{\otimes}\ell_2)]^{\widehat{\otimes} n} = 1,$$

and

$$db_w[\mathcal{B}_+\widehat{\otimes}\mathcal{K}(\ell_2\widehat{\otimes}\ell_2)_+]^{\widehat{\otimes} n} = 2n.$$

Higher-dimensional amenability and the cyclic cohomology of Banach algebras

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Contents

Amenability

B. E. Johnson (1972): $\mathcal{A}$ is **amenable**
$\iff$ there exists a **virtual diagonal** $M \in (\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ such that for all $a \in \mathcal{A}$,

$$aM = Ma, \;\; \pi^{**}(M)a = a.$$

Here π is the product map on $\mathcal{A}$.

n-virtual diagonals

n-virtual diagonals and higher-dimensional amenability of Banach algebras were investigated by E.G. Effros and A. Kishimoto ([EfKi87]) for unital algebras and by A.L.T. Paterson and R.R. Smith ([Pa96], [PaSm97]) in the non-unital case.

Recall [He89] that

$$\mathcal{H}^n(\mathcal{A}, X) \cong \mathrm{Ext}^n_{\mathcal{A}^e}(\mathcal{A}_+, X),$$

here $\mathcal{A}^e = \mathcal{A}_+ \hat{\otimes} \mathcal{A}_+^{op}$ is the enveloping algebra of $\mathcal{A}$, $\mathcal{A}_+^{op}$ is the opposite algebra of $\mathcal{A}_+$ with multiplication $a \cdot b = ba$.

To calculate $\mathrm{Ext}^n_{\mathcal{A}^e}(\mathcal{A}_+, X)$ one can take, for the $\mathcal{A}$-bimodule $\mathcal{A}_+$, the following admissible projective resolution in $\mathcal{A}$-mod-$\mathcal{A}$

$$0 \longleftarrow \mathcal{A}_+ \xleftarrow{\pi_2} \mathcal{A}_+ \hat{\otimes} \mathcal{A}_+ \longleftarrow \ldots \mathcal{A}_+ \hat{\otimes} \mathcal{A}^{\hat{\otimes}(n-3)} \hat{\otimes} \mathcal{A}_+ \xleftarrow{\pi_n} \mathcal{A}_+ \hat{\otimes} \mathcal{A}^{\hat{\otimes}(n-2)} \hat{\otimes} \mathcal{A}_+ \longleftarrow \ldots,$$

where the differentials π_n are given by

$$\pi_2(\alpha_1 \otimes \alpha_2) = \alpha_1 \alpha_2,$$

for $n \geq 3$,

$$\pi_n(\alpha_1 \otimes a_2 \otimes a_3 \ldots a_{n-1} \otimes \alpha_n) = \alpha_1 a_2 \otimes a_3 \otimes \ldots a_{n-1} \otimes \alpha_n +$$

$$\sum_{i=2}^{n-2} (-1)^i (\alpha_1 \otimes a_2 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes \alpha_n) + (-1)^{n-1} \alpha_1 \otimes a_2 \otimes \cdots \otimes a_{n-1} \alpha_n.$$

n-virtual diagonals - continued

Definition 4. [Pa96] *Let $n \geq 2$. An n-virtual diagonal for a Banach algebra $\mathcal{A}$ is an $(n-1)$-cocycle* (that is, $d^{n-1}(D) = 0$)

$$D : \mathcal{A}^{\hat{\otimes}(n-1)} \to \left(\mathcal{A}_+ \hat{\otimes} \mathcal{A}^{\hat{\otimes}(n-1)} \hat{\otimes} \mathcal{A}_+\right)^{**}$$

such that for all $w \in \mathcal{A}^{\hat{\otimes}(n-1)}$,

$$\pi_{n+1}^{**}(D(w)) = \pi_{n+1}(e \otimes w \otimes e).$$

n-virtual diagonals - continued

Theorem 11. ([Pa96]) *Let $n \geq 2$. The following conditions are equivalent for a Banach algebra $\mathcal{A}$:*

(i) *$\mathcal{A}$ is n-amenable;*

(ii) $\mathrm{Ker}\ \pi_n$ *is flat in $\mathcal{A}$-mod-$\mathcal{A}$;*

(iii) *the sequence*

$$0 \longrightarrow (\mathrm{Ker}\ \pi_n)^* \stackrel{(\pi_{n+1})^*}{\longrightarrow} (\mathcal{A}_+ \hat{\otimes} \mathcal{A}^{\hat{\otimes}(n-1)} \hat{\otimes} \mathcal{A}_+)^* \longrightarrow (\mathrm{Ker}\ \pi_{n+1})^* \longrightarrow 0$$

splits in $\mathcal{A}$-mod-$\mathcal{A}$; that is, there exists a morphism of $\mathcal{A}$-bimodules

$$\rho : (\mathcal{A}_+ \hat{\otimes} \mathcal{A}^{\hat{\otimes}(n-1)} \hat{\otimes} \mathcal{A}_+)^* \to (\mathrm{Ker}\ \pi_n)^*$$

such that $\rho \circ (\pi_{n+1})^* = \mathrm{id}_{(\mathrm{Ker}\ \pi_\mathrm{n})^*}$.

(iv) *there exists an n-virtual diagonal for $\mathcal{A}$.*

Note, if ρ from (iii) exists then $D : \mathcal{A}^{\hat{\otimes}(n-1)} \to \left(\mathcal{A}_+ \hat{\otimes} \mathcal{A}^{\hat{\otimes}(n-1)} \hat{\otimes} \mathcal{A}_+\right)^{**}$ can be defined by

$$D(w) = \rho^*(\pi_{n+1}(e \otimes w \otimes e)).$$

Theorem 12. ([He89]) *The following conditions are equivalent for a Banach algebra $\mathcal{A}$*

(i) $\mathcal{A}$ *is amenable;*

(ii) $\mathrm{Ker}\ \pi_1 = \mathcal{A}_+$ *is flat in $\mathcal{A}$-mod-$\mathcal{A}$;*

(iii) *the sequence*

$$0 \longrightarrow (\mathcal{A}_+)^* \xrightarrow{(\pi_2)^*} (\mathcal{A}_+ \hat{\otimes} \mathcal{A}_+)^* \longrightarrow (\mathrm{Ker}\ \pi_2)^* \longrightarrow 0$$

splits in $\mathcal{A}$-mod-$\mathcal{A}$; that is, there exists a morphism of $\mathcal{A}$-bimodules

$$\rho : (\mathcal{A}_+ \hat{\otimes} \mathcal{A}_+)^* \to (\mathcal{A}_+)^*$$

such that $\rho \circ (\pi_2)^* = \mathrm{id}_{\ (\mathcal{A}_+)^*}$.

T_n and 2-virtial diagonal

Theorem 13. [PaSm97; Theorem 4.1] *Let $n \geq 2$. The algebra T_n of upper triangular $n \times n$-complex matrices is 2-amenable, but not amenable.*

There exists a 2-virtial diagonal

$$D : T_n \to T_n \hat{\otimes} T_n \hat{\otimes} T_n$$

which is defined by

$$D(e_{ij})) = \sum_{s \neq j} e_{ij} \otimes e_{jj} \otimes e_{ss} - \sum_{t \neq i} e_{tt} \otimes e_{tt} \otimes e_{ij} + \sum_{p=i}^{j-1} e_{ip} \otimes e_{p(p+1)} \otimes e_{(p+1)j}$$

where e_{ij}, $1 \leq i \leq j \leq n$, is the canonical basis of matrix units for T_n. The 3rd sum is 0 if $i = j$.

Check that D is a derivation on T_n and that, for all i, j,

$$\pi_3(D(e_{ij})) = e_{ij} \otimes e - e \otimes e_{ij}.$$

External products of Hochschild cohomology of Banach algebras with b.a.i.

Theorem 14. [Ly12] *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras with b.a.i., let X be an essential Banach $\mathcal{A}$-bimodule and let Y be an essential Banach $\mathcal{B}$-bimodule. Then for $n \geq 0$, up to topological isomorphism,*

$$\mathcal{H}^n(\mathcal{A}\widehat{\otimes}\mathcal{B}, (X\widehat{\otimes}Y)^*) = H^n((\mathcal{C}_\sim(\mathcal{A}, X)\widehat{\otimes}\mathcal{C}_\sim(\mathcal{B}, Y))^*).$$

Proof. Consider the flat pseudo-resolutions $\beta(\mathcal{A})$ and $\beta(\mathcal{B})$ of $\mathcal{A}$ and $\mathcal{B}$ in the categories of bimodules. One can show that their projective tensor product $\beta(\mathcal{A})\hat{\otimes}\beta(\mathcal{B})$ is an $\mathcal{A}\hat{\otimes}\mathcal{B}$-flat pseudo-resolution of $\mathcal{A}\hat{\otimes}\mathcal{B}$ in $\mathcal{A}\hat{\otimes}\mathcal{B}\text{-mod-}\mathcal{A}\hat{\otimes}\mathcal{B}$.

It can be proved that, up to topological isomorphism,

$$\mathcal{H}^n(\mathcal{A}\hat{\otimes}\mathcal{B}, (X\hat{\otimes}Y)^*) =$$

$$H^n(((X\hat{\otimes}Y)\hat{\otimes}_{(\mathcal{A}\hat{\otimes}\mathcal{B})^e}(\beta(\mathcal{A})\hat{\otimes}\beta(\mathcal{B})))^*) = H^n((\mathcal{C}_\sim(\mathcal{A}, X)\hat{\otimes}\mathcal{C}_\sim(\mathcal{B}, Y))^*).$$

Theorem 15. [Ly12] *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras with b.a.i., let X be an essential Banach $\mathcal{A}$-bimodule and let Y be an essential Banach $\mathcal{B}$-bimodule. Suppose $\mathcal{A}$ is amenable. Then, for $n \geq 0$, up to topological isomorphism,*

$$\mathcal{H}^n(\mathcal{A} \widehat{\otimes} \mathcal{B}, (X \widehat{\otimes} Y)^*) = \mathcal{H}^n(\mathcal{B}, (X/[X, \mathcal{A}] \widehat{\otimes} Y)^*),$$

where $b \cdot (\bar{x} \otimes y) = (\bar{x} \otimes b \cdot y)$ and $(\bar{x} \otimes y) \cdot b = (\bar{x} \otimes y \cdot b)$ for $\bar{x} \in X/[X, \mathcal{A}]$, $y \in Y$ and $b \in \mathcal{B}$.

To calculate $\mathcal{H}^n(\mathcal{A} \widehat{\otimes} \mathcal{B}, (X \widehat{\otimes} Y)^*)$ we need a Künneth formula in topological homology. The following proof of the Künneth formula is from the joint work with F. Gourdeau and M.C. White [GLW05, GLW04].

The homology groups of a chain complex $\mathcal{X}$

A chain complex $\mathcal{X}$ in $\mathcal{F}r$ (in $\mathcal{B}an$) is a family of Fréchet (Banach) spaces X_n and continuous linear maps d_n (called boundary maps)

$$\dots \xleftarrow{d_{n-2}} X_{n-1} \xleftarrow{d_{n-1}} X_n \xleftarrow{d_n} X_{n+1} \xleftarrow{d_{n+1}} \dots$$

such that $\mathrm{Im}\, d_n \subset \mathrm{Ker}\, d_{n-1}$.

The subspace $\mathrm{Im}\, d_n$ of X_n is denoted by $B_n(\mathcal{X})$ and its elements are called *boundaries*.

The Fréchet (Banach) subspace $\mathrm{Ker}\, d_{n-1}$ of X_n is denoted by $Z_n(\mathcal{X})$ and its elements are *cycles*.

The *homology groups of* $\mathcal{X}$ are defined by

$$H_n(\mathcal{X}) = Z_n(\mathcal{X})/B_n(\mathcal{X}).$$

A chain complex $\mathcal{X}$ is called *bounded* if $X_n = \{0\}$ whenever n is less than a certain fixed integer $N \in \mathbb{Z}$.

$H^n(\mathcal{X}^*) \cong H_n(\mathcal{X})^*$ for $\mathcal{X}$ such that all boundary maps have closed range

Proposition 8. [GLW05] *Let $\mathcal{X}$ be a chain complex of Fréchet (Banach) spaces and $\mathcal{X}^*$ the strong dual cochain complex. Then the following are equivalent:*
(1) $H_n(\mathcal{X}) = \mathrm{Ker}\ d_{n-1}/\mathrm{Im}\ d_n$ *is a Fréchet (Banach) space;*
(2) $B_n(\mathcal{X}) = \mathrm{Im}\ d_n$ *is closed in* X_n*;*
(3) d_n *has closed range;*
(4) *the dual map* $d^n = d_n^*$ *has closed range;*
(5) $B^{n+1}(\mathcal{X}^*) = \mathrm{Im}\ d_n^*$ *is strongly closed in* $(X_{n+1})^*$*;*

In the category of Banach spaces, they are equivalent to:
(6) $B^{n+1}(\mathcal{X}^*)$ *is a Banach space;*
(7) $H^{n+1}(\mathcal{X}^*) = \mathrm{Ker}\ d_{n+1}^*/\mathrm{Im}\ d_n^*$ *is a Banach space.*

Moreover, whenever $H_n(\mathcal{X})$ and $H^n(\mathcal{X}^*)$ are Banach spaces, then

$$H^n(\mathcal{X}^*) \cong H_n(\mathcal{X})^*.$$

Strictly flat Banach spaces

Definition 5. *A Fréchet (Banach) space G is strictly flat if for every short exact sequence of Fréchet (Banach) spaces and continuous linear operators $0 \to X \to Y \to Z \to 0$, the short exact sequence*

$$0 \to G\hat{\otimes}X \to G\hat{\otimes}Y \to G\hat{\otimes}Z \to 0$$

is also exact.

Example 4. (i) *Nuclear Fréchet spaces are strictly flat in $\mathcal{F}r$.* (ii) *Finite-dimensional Banach spaces and $L^1(\Omega, \mu)$ are strictly flat in $\mathcal{B}an$.*

Lemma 3. *Let*

$$0 \to Y \xrightarrow{i} Z \xrightarrow{j} W \to 0 \tag{6}$$

be a short exact sequence of Banach spaces and continuous linear operators. Suppose W is strictly flat in $\mathcal{B}an$. Then the sequence (6) is weakly admissible and therefore is topologically pure in $\mathcal{B}an$.

A Künneth formula for $G \hat{\otimes} \mathcal{X}$

Proposition 9. *Let G be a Fréchet (Banach) space, let $\mathcal{X}$ be a chain complex in $\mathcal{F}r$ (in $\mathcal{B}an$) such that all boundary maps d have closed range, let $G \hat{\otimes} \mathcal{X}$ be the chain complex with boundary maps $1_G \otimes d$, and let $n \in \mathbb{Z}$. Suppose, for $k = n-1$ and for $k = n$, the following exact sequences of Fréchet (Banach) spaces*

$$0 \to Z_k(\mathcal{X}) \stackrel{i_k}{\to} X_k \stackrel{\tilde{d}_{k-1}}{\to} B_{k-1}(\mathcal{X}) \to 0 \tag{7}$$

and

$$0 \to B_k(\mathcal{X}) \stackrel{j_k}{\to} Z_k(\mathcal{X}) \stackrel{\sigma_k}{\to} H_k(\mathcal{X}) \to 0 \tag{8}$$

are topologically pure in $\mathcal{F}r$ (in $\mathcal{B}an$). Then the natural inclusions induce topological isomorphisms:
(i) $G \hat{\otimes} Z_n(\mathcal{X}) \cong Z_n(G \hat{\otimes} \mathcal{X})$,
(ii) $G \hat{\otimes} B_n(\mathcal{X}) \cong B_n(G \hat{\otimes} \mathcal{X})$,

(iii)

$$\begin{aligned} G\hat{\otimes}H_n(\mathcal{X}) &\cong G\hat{\otimes}Z_n(\mathcal{X})/(1_G \otimes j_n)(G\hat{\otimes}B_n(\mathcal{X})) \\ &\cong Z_n(G\hat{\otimes}\mathcal{X})/B_n(G\hat{\otimes}\mathcal{X}) \\ &= H_n(G\hat{\otimes}\mathcal{X}). \end{aligned}$$

In particular, the boundary map $1_G \otimes d_n$ *also has closed range.*

A Künneth formula in topological homology

Theorem 16. [GLW05] *Let $\mathcal{X}$ and $\mathcal{Y}$ be bounded chain complexes in $\mathcal{F}r$ (in $\mathcal{B}an$) such that all boundary maps have closed range. Suppose that the following exact sequences of Fréchet (Banach) spaces are topologically pure for all n:*

$$0 \to Z_n(\mathcal{X}) \stackrel{i}{\to} X_n \stackrel{\tilde{d}_{\mathcal{X}}}{\to} B_{n-1}(\mathcal{X}) \to 0 \tag{9}$$

and

$$0 \to B_n(\mathcal{X}) \stackrel{j}{\to} Z_n(\mathcal{X}) \stackrel{\sigma}{\to} H_n(\mathcal{X}) \to 0. \tag{10}$$

Suppose also that one of the following two cases is satisfied.

Case 1. *The following exact sequences of Fréchet (Banach) spaces are topologically pure for all n:*

$$0 \to Z_n(\mathcal{Y}) \stackrel{i}{\to} Y_n \xrightarrow{\tilde{d}_{\mathcal{Y}}} B_{n-1}(\mathcal{Y}) \to 0 \tag{11}$$

and

$$0 \to B_n(\mathcal{Y}) \stackrel{j}{\to} Z_n(\mathcal{Y}) \stackrel{\sigma}{\to} H_n(\mathcal{Y}) \to 0. \tag{12}$$

Case 2. *$Z_n(\mathcal{X})$ and $B_n(\mathcal{X})$ are strictly flat for all n.*

Then, up to topological isomorphism,

$$\bigoplus_{m+q=n} [H_m(\mathcal{X})\hat{\otimes} H_q(\mathcal{Y})] = H_n(\mathcal{X}\hat{\otimes}\mathcal{Y}).$$

Corollary 2. *(M. Karoubi). Let $\mathcal{X}$ and $\mathcal{Y}$ be bounded chain complexes of Fréchet spaces and continuous operators such that all boundary maps have closed range. Suppose that $\mathcal{X}$ or $\mathcal{Y}$ is a complex of nuclear Fréchet spaces. Then, up to topological isomorphism,*

$$\bigoplus_{m+q=n} [H_m(\mathcal{X})\hat{\otimes} H_q(\mathcal{Y})] = H_n(\mathcal{X}\hat{\otimes}\mathcal{Y}).$$

Proof. For all n, the short exact sequences of nuclear Fréchet spaces (9), (10) are topologically pure in $\mathcal{F}r$. Nuclear Fréchet spaces $B_n(\mathcal{X})$ and $Z_n(\mathcal{X})$ are strictly flat in $\mathcal{F}r$. The result follows from Theorem 16 (Case 2).

Corollary 3. *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be bounded chain complexes in* $\mathcal{B}an$ *such that all boundary maps have closed range. Suppose that, for all n,* $B_n(\mathcal{X})$ *and* $H_n(\mathcal{X})$ *are strictly flat. Then, up to topological isomorphism,*

$$\bigoplus_{m+q=n} [H_m(\mathcal{X})\hat{\otimes}H_q(\mathcal{Y})] = H_n(\mathcal{X}\hat{\otimes}\mathcal{Y}).$$

Proof. It is known that if $B_n(\mathcal{X})$ and $H_n(\mathcal{X})$ are strictly flat then $Z_n(\mathcal{X})$ is strictly flat too. By Lemma 3, strict flatness of $B_{n-1}(\mathcal{X})$ and $H_n(\mathcal{X})$ implies that the short exact sequences of Banach spaces (9), (10) are topologically pure in $\mathcal{B}an$. The result follows from Theorem 16 (Case 2).

A Künneth formula for $\mathcal{H}_n(\mathcal{A}\hat{\otimes}\mathcal{B}, X\hat{\otimes}Y)$

Theorem 17. [GLW04] *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras with bounded approximate identities, let X be an essential Banach $\mathcal{A}$-bimodule and let Y be an essential Banach $\mathcal{B}$-bimodule. Suppose that all boundary maps of the standard homology complexes $\mathcal{C}_\sim(\mathcal{A}, X)$ and $\mathcal{C}_\sim(\mathcal{B}, Y)$ have closed range and, for all n, $B_n(\mathcal{A}, X)$ and $\mathcal{H}_n(\mathcal{A}, X)$ are strictly flat. Then, for $n \geq 0$*

$$\mathcal{H}_n(\mathcal{A}\hat{\otimes}\mathcal{B}, X\hat{\otimes}Y) \cong \bigoplus_{m+q=n} \mathcal{H}_m(\mathcal{A}, X)\hat{\otimes}\mathcal{H}_q(\mathcal{B}, Y),$$

and

$$\mathcal{H}^n(\mathcal{A}\hat{\otimes}\mathcal{B}, (X\hat{\otimes}Y)^*) \cong \bigoplus_{m+q=n} [\mathcal{H}_m(\mathcal{A}, X)\hat{\otimes}\mathcal{H}_q(\mathcal{B}, Y)]^*.$$

The simplicial cohomology of $\ell^1(\mathbb{Z}_+^k)$

Let $\mathcal{A} = \ell^1(\mathbb{Z}_+)$ where

$$\ell^1(\mathbb{Z}_+) = \left\{ (a_n)_{n=0}^{\infty} : \sum_{n=0}^{\infty} |a_n| < \infty \right\}$$

is the unital semigroup Banach algebra of $\mathbb{Z}_+$ with convolution multiplication and norm $\|(a_n)_{n=0}^{\infty}\| = \sum_{n=0}^{\infty} |a_n|$. Recall that $\ell^1(\mathbb{Z}_+)$ is isometrically isomorphic to the unital commutative Banach algebra

$$A^+(\overline{\mathbf{D}}) = \left\{ f = \sum_{n=0}^{\infty} a_n z^n : \sum_{n=0}^{\infty} |a_n| < \infty \right\}$$

of absolutely convergent Taylor series on $\overline{\mathbf{D}}$ with pointwise multiplication and norm $\|f\| = \sum_{n=0}^{\infty} |a_n|$, where $\overline{\mathbf{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$ is the closed disc.

Let $\mathcal{I} = \ell^1(\mathbb{N})$ be the closed ideal of $\ell^1(\mathbb{Z}_+)$ consisting of those elements with $a_0 = 0$.

Theorem 18. [GLW05] *Let $\mathcal{A} = \ell^1(\mathbb{Z}_+)$. Then, up to topological isomorphism,*

i. $\mathcal{H}_n(\ell^1(\mathbb{Z}_+^k), \ell^1(\mathbb{Z}_+^k)) = 0$ *if* $n > k$,

ii. $\mathcal{H}_n(\ell^1(\mathbb{Z}_+^k), \ell^1(\mathbb{Z}_+^k)) = \bigoplus^{\binom{k}{n}} \left(\mathcal{I}^{\hat{\otimes}^n} \hat{\otimes} \mathcal{A}^{\hat{\otimes}^{k-n}}\right)$ *if* $n \leq k$,

iii. $\mathcal{H}^n(\ell^1(\mathbb{Z}_+^k), \ell^1(\mathbb{Z}_+^k)^*) = 0$ *if* $n > k$ *and*

iv. $\mathcal{H}^n(\ell^1(\mathbb{Z}_+^k), \ell^1(\mathbb{Z}_+^k)^*) = \bigoplus^{\binom{k}{n}} \left[\left(\mathcal{I}^{\hat{\otimes}^n} \hat{\otimes} \mathcal{A}^{\hat{\otimes}^{k-n}}\right)^*\right]$ *if* $n \leq k$.

The simplicial cohomology of $L^1(\mathbb{R}_+^k)$

Let $L^1(\mathbb{R}_+)$ be the convolution Banach function algebra of complex-valued, Lebesgue measurable functions f on $\mathbb{R}_+$ with finite L^1-norm (that is such that $\int_0^\infty |f(t)|dt < \infty$). For $f, g \in L^1(\mathbb{R}_+)$, $f * g(t) = \int_0^\infty f(t-s)g(s)ds$, $t \in \mathbb{R}_+$.

Theorem 19. [GLW04] *The following results hold:*

i. $\mathcal{H}_n(L^1(\mathbb{R}_+^k), L^1(\mathbb{R}_+^k)) = 0$ *if* $n > k$*;*

ii. $\mathcal{H}_n(L^1(\mathbb{R}_+^k), L^1(\mathbb{R}_+^k)) \cong \bigoplus^{\binom{k}{n}} L^1(\mathbb{R}_+^k)$ *if* $n \leq k$*;*

iii. $\mathcal{H}^n(L^1(\mathbb{R}_+^k), L^1(\mathbb{R}_+^k)^*) = 0$ *if* $n > k$*; and*

iv. $\mathcal{H}^n(L^1(\mathbb{R}_+^k), L^1(\mathbb{R}_+^k)^*) \cong \bigoplus^{\binom{k}{n}} L^\infty(\mathbb{R}_+^k)$ *if* $n \leq k$*.*

The simplicial cohomology of $L^1(\mathbf{R}_+^k) \widehat{\otimes} \mathcal{C}$

Theorem 20. [GLW04], [Ly12] *Let $\mathcal{C}$ be an amenable Banach algebra. Then*

$$\mathcal{H}_n(L^1(\mathbf{R}_+^k) \widehat{\otimes} \mathcal{C}, L^1(\mathbf{R}_+^k) \widehat{\otimes} \mathcal{C}) \cong \{0\} \text{ if } \ n > k;$$

$$\mathcal{H}^n\left(L^1(\mathbf{R}_+^k) \widehat{\otimes} \mathcal{C}, \left(L^1(\mathbf{R}_+^k) \widehat{\otimes} \mathcal{C}\right)^*\right) \cong \{0\} \text{ if } \ n > k;$$

up to topological isomorphism,

$$\mathcal{H}_n(L^1(\mathbf{R}_+^k) \widehat{\otimes} \mathcal{C}, L^1(\mathbf{R}_+^k) \widehat{\otimes} \mathcal{C}) \cong \bigoplus^{\binom{k}{n}} L^1(\mathbf{R}_+^k) \widehat{\otimes} (\mathcal{C}/[\mathcal{C}, \mathcal{C}]) \text{ if } n \leq k;$$

and

$$\mathcal{H}^n(L^1(\mathbf{R}_+^k) \widehat{\otimes} \mathcal{C}, (L^1(\mathbf{R}_+^k) \widehat{\otimes} \mathcal{C})^*) \cong \bigoplus^{\binom{k}{n}} \left[L^1(\mathbf{R}_+^k) \widehat{\otimes} (\mathcal{C}/[\mathcal{C}, \mathcal{C}])\right]^*$$

if $n \leq k$.

Mixed complexes in the category of Fréchet spaces.

We shall use an extension of the mixed complex approach to continuous cyclic type theories in the category $\mathcal{F}r$ of Fréchet spaces and continuous linear operators. (See C. Kassel, 1987; J. Cuntz and D. Quillen, 1995.)

A **mixed complex** $(\mathcal{M}, b, B)$ in the category $\mathcal{F}r$ is a family $\mathcal{M} = \{M_n\}$ of Fréchet spaces M_n equipped with continuous linear operators $b_n : M_n \to M_{n-1}$ and $B_n : M_n \to M_{n+1}$, which satisfy the identities

$$b^2 = bB + Bb = B^2 = 0.$$

We assume that in degree zero the differential b is identically equal to zero.

We arrange the mixed complex $(\mathcal{M}, b, B)$ in the double complex

$$
\begin{array}{ccccccccccc}
\dots & & \dots & & \dots & & \dots & & \dots & & \dots \\
b\downarrow & & b\downarrow & & b\downarrow & & b\downarrow & & b\downarrow & & b\downarrow \\
M_4 & \stackrel{B}{\leftarrow} & M_3 & \stackrel{B}{\leftarrow} & M_2 & \stackrel{B}{\leftarrow} & M_1 & \stackrel{B}{\leftarrow} & M_0 & \stackrel{B}{\leftarrow} & 0 \\
b\downarrow & & b\downarrow & & b\downarrow & & b\downarrow & & b\downarrow & & \\
M_3 & \stackrel{B}{\leftarrow} & M_2 & \stackrel{B}{\leftarrow} & M_1 & \stackrel{B}{\leftarrow} & M_0 & \stackrel{B}{\leftarrow} & 0 & & \\
b\downarrow & & b\downarrow & & b\downarrow & & b\downarrow & & & & \\
M_2 & \stackrel{B}{\leftarrow} & M_1 & \stackrel{B}{\leftarrow} & M_0 & \stackrel{B}{\leftarrow} & 0 & & & & \\
b\downarrow & & b\downarrow & & b\downarrow & & & & & & \\
M_1 & \stackrel{B}{\leftarrow} & M_0 & \stackrel{B}{\leftarrow} & 0 & & & & & & \\
b\downarrow & & b\downarrow & & & & & & & & \\
M_0 & \stackrel{B}{\leftarrow} & 0 & & & & & & & & \\
b\downarrow & & & & & & & & & & \\
0 & & & & & & & & & &
\end{array}
$$

Homology of mixed complexes

There are three homology theories that can be naturally associated with a mixed complex.

The **Hochschild homology** $H_*^b(\mathcal{M})$ of $(\mathcal{M}, b, B)$ is the homology of the chain complex $(\mathcal{M}, b)$, that is,

$$H_n^b(\mathcal{M}) = H_n(\mathcal{M}, b)$$
$$= \mathrm{Ker}\,\{b_n : M_n \to M_{n-1}\}/\mathrm{Im}\,\{b_{n+1} : M_{n+1} \to M_n\}.$$

To define the cyclic homology of $(\mathcal{M}, b, B)$, let us denote by $\mathcal{B}_c\mathcal{M}$ the total complex of the above double complex, that is,

$$\to (\mathcal{B}_c\mathcal{M})_n \overset{b+B}{\to} (\mathcal{B}_c\mathcal{M})_{n-1} \to \cdots \overset{b+B}{\to} (\mathcal{B}_c\mathcal{M})_0 \to 0,$$

where the spaces

$$(\mathcal{B}_c\mathcal{M})_0 = M_0, \ \ldots,$$
$$(\mathcal{B}_c\mathcal{M})_{2k-1} = M_1 \oplus M_3 \oplus \cdots \oplus M_{2k-1}$$

and

$$(\mathcal{B}_c\mathcal{M})_{2k} = M_0 \oplus M_2 \oplus \cdots \oplus M_{2k}.$$

are equipped with the product topology, and continuous linear operators $b + B$ are defined by

$$(b + B)(y_0, \ldots, y_{2k}) = (by_2 + By_0, \ldots, by_{2k} + By_{2k-2})$$

and

$$(b + B)(y_1, \ldots, y_{2k+1}) = (by_1, \ldots, by_{2k+1} + By_{2k-1}).$$

The **cyclic homology** of $(\mathcal{M}, b, B)$ is by definition

$$H^c_*(\mathcal{M}, b, B) = H_*(\mathcal{B}_c\mathcal{M}, b + B).$$

The periodic cyclic homology of $(\mathcal{M}, b, B)$ is defined in terms of the complex

$$\to (\mathcal{B}_p\mathcal{M})_{ev} \overset{b+B}{\to} (\mathcal{B}_p\mathcal{M})_{odd} \overset{b+B}{\to} (\mathcal{B}_p\mathcal{M})_{ev} \overset{b+B}{\to} (\mathcal{B}_p\mathcal{M})_{odd} \to,$$

where even/odd chains are elements of the product spaces

$$(\mathcal{B}_p\mathcal{M})_{ev} = \prod_{n\geq 0} M_{2n}$$

and

$$(\mathcal{B}_p\mathcal{M})_{odd} = \prod_{n\geq 0} M_{2n+1},$$

respectively. The spaces $(\mathcal{B}_p\mathcal{M})_{ev/odd}$ are Fréchet spaces with respect to the product topology. The continuous differential $b + B$ is defined as an obvious extension of the above. The **periodic cyclic homology** of $(\mathcal{M}, b, B)$ is

$$H^p_\nu(\mathcal{M}, b, B) = H_\nu(\mathcal{B}_p\mathcal{M}, b + B),$$

where $\nu \in \mathbf{Z}/2\mathbf{Z}$.

The three different homologies of a mixed complex are related in the following interesting way.

Lemma 4. . *Let* $(\mathcal{M}, b, B)$ *be a mixed complex with*

$$H_n^b(\mathcal{M}) = 0 \quad \text{for all } n \geq 0.$$

Then also

$$H_n^c(\mathcal{M}, b, B) = H_n^p(\mathcal{M}, b, B) = 0 \text{ for} \quad \text{all } n \geq 0.$$

The Connes-Tsygan long exact sequence associated with $(\mathcal{M}, b, B)$

First we define the Connes periodicity operator

$$S : (\mathcal{B}_c\mathcal{M})_n \to (\mathcal{B}_c\mathcal{M})_{n-2}$$

by the formulae

$$S(y_0, \ldots, y_{2n-2}, y_{2n}) = (y_0, \ldots, y_{2n-2})$$

and

$$S(y_1, \ldots, y_{2n+1}) = (y_1, \ldots, y_{2n-1})$$

for $n \geq 2$. We put $SM_0 = 0$, $SM_1 = 0$.

We then have the following short exact sequence of complexes of Fréchet spaces:

$$0 \to (\mathcal{M}, b) \xrightarrow{I} (\mathcal{B}_c\mathcal{M}, b + B) \xrightarrow{S} (\mathcal{B}_c\mathcal{M}[2], (b + B)[2]) \to 0,$$

where I is the natural inclusion.

Hence there is a long exact sequence

$$\to H^b_{n+2}(\mathcal{M}) \to H^c_{n+2}(\mathcal{M}, b, B) \to H^c_n(\mathcal{M}, b, B) \to H^b_{n+1}(\mathcal{M}) \to$$

of homology groups with continuous induced maps.

Here, for a chain complex (K, d), the complex $(K[m], d[m])$ is defined by $(K[m])_q = K_{q-m}$ and $(d[m])_q = (-1)^m d_{q-m}$.

Hochschild homology for nonunital Fréchet algebras

Let $\mathcal{A}$ be a Fréchet algebra, not necessarily unital. The **continuous bar** and **'naive' Hochschild homology of** $\mathcal{A}$ are defined respectively as

$$\mathcal{H}_*^{bar}(\mathcal{A}) = H_*(\mathcal{C}(\mathcal{A}), b')$$

and

$$\mathcal{H}_*^{naive}(\mathcal{A}) = H_*(\mathcal{C}(\mathcal{A}), b),$$

where

$$\mathcal{C}_n(\mathcal{A}) = \mathcal{A}^{\hat{\otimes}(n+1)},$$

and the differentials b, b' are given by

$$b'(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i (a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n)$$

and

$$b(a_0 \otimes \cdots \otimes a_n) = b'(a_0 \otimes \cdots \otimes a_n) + (-1)^n (a_n a_0 \otimes \cdots \otimes a_{n-1}).$$

Note that

- $H_*^{naive}(\mathcal{A}) = \mathcal{H}_*(\mathcal{A}, \mathcal{A})$, the continuous homology of $\mathcal{A}$ with coefficients in $\mathcal{A}$.
- $H_n^{bar}(\mathcal{A}) = \mathcal{H}_{n+1}(\mathcal{A}, \mathbb{C})$, for the trivial A-bimodule $\mathbb{C}$.

Cyclic type homology for nonunital Fréchet algebras

Consider the **mixed complex**

$$(\bar{\Omega}\mathcal{A}_+, \tilde{b}, \tilde{B})$$

in $\mathcal{F}r$, where

$$\bar{\Omega}^n\mathcal{A}_+ = \mathcal{A}^{\hat{\otimes}(n+1)} \oplus \mathcal{A}^{\hat{\otimes}n},$$

$\hat{\otimes}$ is the completed projective tensor product, and

$$\tilde{b} = \begin{pmatrix} b & 1-\lambda \\ 0 & -b' \end{pmatrix}; \quad \tilde{B} = \begin{pmatrix} 0 & 0 \\ N & 0 \end{pmatrix}$$

with

$$\lambda(a_1 \otimes \cdots \otimes a_n) = (-1)^{n-1}(a_n \otimes a_1 \otimes \cdots \otimes a_{n-1})$$

and

$$N = \mathrm{id} + \lambda + \cdots + \lambda^{n-1}.$$

(See Loday's book.)

The mixed complex $(\bar{\Omega}A_+, \tilde{b}, \tilde{B})$ is the following

$$
\begin{array}{ccccccccc}
\dots & & \dots & & \dots & & \dots & & \dots \\
\tilde{b}\downarrow & & \tilde{b}\downarrow & & \tilde{b}\downarrow & & \tilde{b}\downarrow & & \tilde{b}\downarrow \\
\mathcal{A}^{\hat{\otimes}4} \oplus \mathcal{A}^{\hat{\otimes}3} & \overset{\tilde{B}}{\leftarrow} & \mathcal{A}^{\hat{\otimes}3} \oplus \mathcal{A}^{\hat{\otimes}2} & \overset{\tilde{B}}{\leftarrow} & \mathcal{A}^{\hat{\otimes}2} \oplus \mathcal{A} & \overset{\tilde{B}}{\leftarrow} & \mathcal{A} & \overset{\tilde{B}}{\leftarrow} & 0 \\
\tilde{b}\downarrow & & \tilde{b}\downarrow & & \tilde{b}\downarrow & & \tilde{b}\downarrow & & \\
\mathcal{A}^{\hat{\otimes}3} \oplus \mathcal{A}^{\hat{\otimes}2} & \overset{\tilde{B}}{\leftarrow} & \mathcal{A}^{\hat{\otimes}2} \oplus \mathcal{A} & \overset{\tilde{B}}{\leftarrow} & \mathcal{A} & \overset{\tilde{B}}{\leftarrow} & 0 & & \\
\tilde{b}\downarrow & & \tilde{b}\downarrow & & \tilde{b}\downarrow & & & & \\
\mathcal{A}^{\hat{\otimes}2} \oplus \mathcal{A} & \overset{\tilde{B}}{\leftarrow} & \mathcal{A} & \overset{\tilde{B}}{\leftarrow} & 0 & & & & \\
\tilde{b}\downarrow & & \tilde{b}\downarrow & & & & & & \\
\mathcal{A} & \overset{\tilde{B}}{\leftarrow} & 0 & & & & & & \\
\tilde{b}\downarrow & & & & & & & & \\
0 & & & & & & & &
\end{array}
$$

The **continuous Hochschild homology of $\mathcal{A}$**, the **continuous cyclic homology of A** and the **continuous periodic cyclic homology of A** are defined by

$$HH_*(\mathcal{A}) = H^b_*(\bar{\Omega}\mathcal{A}_+, \tilde{b}, \tilde{B}),$$

$$HC_*(\mathcal{A}) = H_*^c(\bar{\Omega}\mathcal{A}_+, \tilde{b}, \tilde{B})$$

and

$$HP_*(\mathcal{A}) = H_*^p(\bar{\Omega}\mathcal{A}_+, \tilde{b}, \tilde{B}).$$

where H_*^b, H_*^c and H_*^p are Hochschild homology, cyclic homology and periodic cyclic homology of the mixed complex $(\bar{\Omega}\mathcal{A}_+, \tilde{b}, \tilde{B})$ in the category $\mathcal{F}r$ of Fréchet spaces and continuous linear operators.

There is also a *cyclic cohomology* theory associated with a Fréchet algebra $\mathcal{A}$, obtained when one replaces the chain complexes of $\mathcal{A}$ by their dual complexes of strong dual spaces (A. Connes, J.-L. Loday).

For example, the continuous bar cohomology $\mathcal{H}^n_{bar}(\mathcal{A})$ of A is the cohomology of the dual complex $(\mathcal{C}(\mathcal{A})^*, (b')^*)$ of $(\mathcal{C}(\mathcal{A}), b')$.

$H_n^{naive}(\mathcal{A})$ and $HH_n(\mathcal{A})$

There exists also the following short exact sequence of complexes of Fréchet spaces

$$0 \to (\mathcal{C}(\mathcal{A}), b) \to (\bar{\Omega}\tilde{A}, \tilde{b}) \to (\mathcal{C}(\mathcal{A})[1], b'[1]) \to 0,$$

which leads to a long exact homology sequence **connecting the three homologies**

$$\to H_n^{bar}(\mathcal{A}) \to H_n^{naive}(\mathcal{A}) \to HH_n(\mathcal{A}) \to H_{n-1}^{bar}(\mathcal{A}) \to .$$

This shows that

$$H_n^{naive}(\mathcal{A}) = HH_n(\mathcal{A}) \quad \text{for all } n \geq 0$$

$\iff$

$$H_n^{bar}(\mathcal{A}) = 0 \quad \text{for all } n \geq 0$$

$\iff$

$$\mathcal{H}_{n+1}(\mathcal{A}, \mathbb{C}) = 0 \quad \text{for all } n \geq 0,$$

where $\mathbb{C}$ is the trivial $\mathcal{A}$-bimodule.

The Connes-Tsygan long exact sequence

We noted above the existence of the Connes-Tsygan long exact sequence for a mixed complex. Thus there is **the Connes-Tsygan sequence for Fréchet algebras**:

$$\to HH_n(\mathcal{A}) \to HC_n(\mathcal{A}) \to HC_{n-2}(\mathcal{A}) \to HH_{n-1}(\mathcal{A}) \to .$$

Theorem 21. *Let A be a Fréchet algebra. Then, for the following statements are equivalent.*

(i) *There exists a long exact Connes-Tsygan sequence of continuous homology groups*

$$\cdots \to H_n^{naive}(A) \to HC_n(A) \to HC_{n-2}(A) \to H_{n-1}^{naive}(A) \to \ldots;$$

(ii) $H_n^{bar}(A) = \{0\}$ *for all* $n \geq 0$*;*

(iii) $H^n_{bar}(A) = \{0\}$ *for all* $n \geq 0$*;*

(iv) *there exists a long exact Connes-Tsygan sequence of continuous cohomology groups*

$$\cdots \to H^n_{naive}(A) \to HC^{n-1}(A) \to HC^{n+1}(A) \to H^{n+1}_{naive}(A) \to \ldots.$$

Examples of Banach algebras $\mathcal{A}$ such that $H_n^{bar}(\mathcal{A}) \neq 0$ for some $n \geq 0$

(i) The maximal ideal

$$\mathcal{A}_0(\mathbf{D}) = \{w : w(0) = 0\}$$

of the disc algebra,

(ii) ℓ_2 with coordinatewise multiplication,

(iii) the algebra $\mathcal{HS}(H)$ of Hilbert-Schmidt operators on an infinite-dimensional Hilbert space H.

See [He92].

Strongly H-unital Banach and Fréchet algebras

A Fréchet algebra A is called **strongly H-unital** if the homology of the complex

$$(\mathcal{C}(A)\hat{\otimes}X, b'\hat{\otimes}\mathrm{id}_X)$$

is trivial for every Fréchet space X. Here id denotes the identity operator.

(M. Wodzicki, 1989)

Strongly H-unital Fréchet algebras:

(i) Unital Fréchet algebras are strongly H-unital. For example, the algebra $O(\mathbf{C})$ of holomorphic functions on $\mathbf{C}$; the algebra $C^\infty(M)$ of smooth functions.

(ii) By extending results of B. E. Johnson one can prove that a Fréchet algebra A with a left or right bounded approximate identity has this property.

(ii.1) It is well-known that any C^*-algebra A has a bounded approximate identity.

(ii.2) The Banach algebra $A = \mathcal{K}(E)$ of compact operators on a Banach space E with the bounded compact approximation property has also a bounded left approximate identity (P. Dixon).

(ii.3) Let $\omega_n(x) = (1+|x|)^{1-\frac{1}{n}}$, $\omega_0(x) = 1+|x|$ and let $A_n =$

$$\left\{ f \in L^1(\mathbf{R}) : \; p_n(f) = \int_{-\infty}^{+\infty} |f(x)|\omega_n(x)dx < \infty \right\},$$

$n = 0, 1, \ldots$. Then A_n is a Banach algebra under the norm p_n. Let

$$A = \bigcap_{n=1}^{\infty} A_n$$

topologised by the collection of norms $p_n, \; n = 0, 1, \ldots$. Then A is a Fréchet algebra with bounded approximate identity (I. G. Craw).

(iii) If A is flat as a right Fréchet A-module and $A^2 = A$, then A is strongly H-unital. Here A^2 is the closed linear span of $\{ab : \; a, b \in A\}$.

For example,

(iii.1) l^1, the Banach algebra ℓ_1 of summable complex sequences (ξ_n) with co-ordinatewise operations (biprojective), (iii.2) the Banach algebra $N(H)$ of nuclear operators in a Hilbert space H (biprojective), and

(iii.3) the algebra $\mathcal{K}(\ell_2 \hat{\otimes} \ell_2)$ of all compact operators on the Banach space $\ell_2 \hat{\otimes} \ell_2$ (a non-amenable, biflat semisimple Banach algebra with a left bounded approximate identity)

are strongly H-unital.

Applications to the cyclic cohomology

Theorem 22. (M. Khalkhali, 1994) *Let $\mathcal{A}$ be a Banach algebra such that for a fixed integer n, $\mathcal{H}^n(\mathcal{A}, X^*) = 0$ for all dual Banach $\mathcal{A}$-bimodules X^*. Then the natural map*

$$HP^*(\mathcal{A}) \to HE^*(\mathcal{A})$$

is an isomorphism.

Applications to the cyclic cohomology - continued

Theorem 23. [Ly06; Corollary 4.1] *Let* $\mathcal{A}$ *be a Fréchet algebra. Suppose that* $db_w\mathcal{A} = m$ *(say,* $\mathcal{A}$ *is* $(m+1)$*-amenable) and* $m = 2L$ *is an even integer. Then,*

(i) *for all* $\ell \geq L$, $HC_{2\ell+2}(\mathcal{A}) = HC_m(\mathcal{A})$ *and* $HC_{2\ell+3}(\mathcal{A}) = HC_{m+1}(\mathcal{A})$*;*

(ii) $HP_0(\mathcal{A}) = HC_m(\mathcal{A})$ *and* $HP_1(\mathcal{A}) = HC_{m+1}(\mathcal{A})$*;*

(iii) *for all* $\ell \geq L$, $HC^{2\ell+2}(\mathcal{A}) = HC^m(\mathcal{A})$ *and* $HC^{2\ell+3}(\mathcal{A}) = HC^{m+1}(\mathcal{A})$*;*

(iv) $HP^0(\mathcal{A}) = HC^m(\mathcal{A})$ *and* $HP^1(\mathcal{A}) = HC^{m+1}(\mathcal{A})$.

(v) *if, furthermore,* $\mathcal{A}$ *is a Banach algebra,*

$HE^0(\mathcal{A}) = HP^0(\mathcal{A}) = HC^m(\mathcal{A})$ *and* $HE^1(\mathcal{A}) = HP^1(\mathcal{A}) = HC^{m+1}(\mathcal{A})$.

There are similar formulae for odd m.

Examples

For these Fréchet algebras: $\mathcal{O}(\mathcal{U})$, $C^\infty(M)$ and $\mathcal{S}(\mathbf{R}^m)$ of rapidly decreasing infinitely smooth functions on $\mathbf{R}^m$, the conditions of Theorem 23 are satisfied.

For example, by Theorem 23 and [Wa; Section VII], for an even m, $HP^0(\mathcal{S}(\mathbf{R}^m)) = HC^m(\mathcal{S}(\mathbf{R}^m))$, the one-dimensional linear space generated by the fundamental m-trace

$$\varphi(f^0, f^1, \dots, f^m) = \int_{\mathbf{R}^m} f^0 df^1 \wedge \cdots \wedge df^m,$$

and $HP^1(\mathcal{S}(\mathbf{R}^m)) = HC^{m+1}(\mathcal{S}(\mathbf{R}^m)) = \{0\}$.

Applications to cyclic cohomology groups - continued

Theorem 24. *Let $\mathcal{A}$ be a biflat Banach algebra. Then*

(i) *for all* $\ell \geq 0$, $HC_{2\ell}(\mathcal{A}) = \mathcal{A}/[\mathcal{A},\mathcal{A}]$ *and* $HC_{2\ell+1}(\mathcal{A}) = \{0\}$;

(ii) $HP_0(\mathcal{A}) = \mathcal{A}/[\mathcal{A},\mathcal{A}]$ *and* $HP_1(\mathcal{A}) = \{0\}$;

(iii) *for all* $\ell \geq 0$, $HC^{2\ell}(\mathcal{A}) = \mathcal{A}^{tr}$ *and* $HC^{2\ell+1}(\mathcal{A}) = \{0\}$;

(iv) $HE^0(\mathcal{A}) = HP^0(\mathcal{A}) = \mathcal{A}^{tr}$ *and* $HE^1(\mathcal{A}) = HP^1(\mathcal{A}) = \{0\}$.

The statement easily follows from previuos results and A. Ya. Helemskii's theorem [He92] on the cyclic cohomology groups of biflat algebras.

Examples: cyclic type cohomology of amenable Banach algebras

(i) Let G be a locally compact group with a left-invariant Haar measure ds. The group algebra $L^1(G)$ of Haar integrable functions on an amenable locally compact group G with convolution product is amenable too. Therefore, by Theorem 24, for the continuous cyclic and periodic cyclic cohomology groups and the entire cyclic cohomology of $L^1(G)$, we have the following. For all $\ell \geq 0$,

$$HE^0(L^1(G)) = HP^0(L^1(G)) = HC^{2\ell}(L^1(G)) = L^1(G)^{tr}$$

where $L^1(G)^{tr} = \{f \in L^\infty(G) : f(ab) = f(ba)$ for all $a, b \in L^1(G)\}$ and

$$HE^1(L^1(G)) = HP^1(L^1(G)) = HC^{2\ell+1}(L^1(G)) = \{0\}.$$

Examples: cyclic type cohomology of amenable Banach algebras - continued

(ii) It is shown in [GJW94] that the Banach algebra $\mathcal{K}(E)$ of compact operators on a Banach space E with property $(\mathbf{A})$ which was defined in [GJW94] is amenable. Property $(\mathbf{A})$ implies that $\mathcal{K}(E)$ contains a bounded sequence of projections of unbounded finite rank, and from this it is easy to show (via embedding of matrix algebras) that there is no non-zero bounded trace on $\mathcal{K}(E)$. Therefore, by Theorem 24, the continuous cyclic and periodic cyclic homology and cohomology groups and the entire cyclic cohomology of $\mathcal{K}(E)$ are trivial, that is, for all $n \geq 0$,

$$HC_n(\mathcal{K}(E)) = \{0\}, \;\; HC^n(\mathcal{K}(E)) = \{0\}$$

and, for $k = 0, 1$,

$$HP_k(\mathcal{K}(E)) = \{0\}, \;\; HE^k(\mathcal{K}(E)) = HP^k(\mathcal{K}(E)) = \{0\}.$$

Examples: cyclic type cohomology of biprojective Banach algebras

(i) The Banach algebra ℓ_1 of summable complex sequences with co-ordinatewise operations is biprojective, and so is biflat. By Theorem 24, for all $k \geq 0$,

$$HP_0(\ell_1) = HC_{2k}(\ell_1) = \ell_1, \;\; HP_1(\ell_1) = HC_{2k+1}(\ell_1) = \{0\},$$

$$HE^0(\ell_1) = HP^0(\ell_1) = HC^{2k}(\ell_1) = \ell_\infty,$$

where ℓ_∞ is the Banach space of bounded sequences, and

$$HE^1(\ell_1) = HP^1(\ell_1) = HC^{2k+1}(\ell_1) = \{0\}.$$

Examples: cyclic type cohomology of biprojective Banach algebras - continued

(ii) The algebra $\mathcal{N}(H)$ of nuclear operators on an infinite-dimensional Hilbert space H with composition multiplication is biprojective, and so biflat. By Theorem 24, for all $k \geq 0$,

$$HP_0(\mathcal{N}(H)) = HC_{2k}(\mathcal{N}(H)) = \mathbb{C}, \;\; HP_1(\mathcal{N}(H)) = HC_{2k+1}(\mathcal{N}(H)) = \{0\},$$

$$HE^0(\mathcal{N}(H)) = HP^0(\mathcal{N}(H)) = HC^{2k}(\mathcal{N}(H)) = \mathbb{C},$$

and

$$HE^1(\mathcal{N}(H)) = HP^1(\mathcal{N}(H)) = HC^{2k+1}(\mathcal{N}(H)) = \{0\}.$$

Thank you

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